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Compact operators on Banach spaces: Fredholm-Riesz

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[Fredholm 1900/1903] treated compact operators as limiting cases of finite-rank operators. [1] [Riesz 1917] defined and made direct use of the compactness condition, more apt for Banach spaces. See [Riesz-Nagy 1952] for extensive discussion in the Hilbert-space situation, and many references to original papers. [Arveson 2002] gives a succinct discussion.

Here, we prove the basic *Fredholm alternative* on Banach spaces, that for compact T and non-zero $\lambda \in \mathbb{C}$, *either* $T - \lambda$ is a bijection, *or* has closed image of codimension equal to the dimension of its kernel. In particular, the only non-zero spectrum is *point* spectrum. [2]

A special case of this is widely useful in *perturbation theory*: it is often important to know, for $\lambda \neq 0$ not an eigenvalue of compact $T : X \to X$, that $T - \lambda$ is *surjective*.

In the appendix, we recall the equivalence of *pre-compactness* and *total boundedness* in complete metric spaces, and recall the Arzela-Ascoli result, that the pre-compact subset of $C^o(X)$ for compact X consist of *equicontinuous, uniformly bounded* collections of functions.

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1. *Compact operators on Banach spaces*

[1.1] Notion of compact operator

A continuous linear operator $T : X \to Y$ on Banach spaces is *compact* when T maps *bounded* sets in X to *pre-compact* sets in Y, that is, sets with compact closure. Since bounded sets lie in some ball in X, and since T is linear, it suffices to verify that T maps the unit ball in X to a pre-compact set in Y.

Finite-rank operators are clearly compact.

Both right and left compositions $S \circ T$ and $T \circ S$ of compact T with continuous S produce compact operators.

Recall a criterion for pre-compactness: a set E in a complete metric space is pre-compact if and only if it is *totally bounded*, in the sense that, give $\varepsilon > 0$, E is covered by finitely-many open balls of radius ε.

We claim that operator-norm limits $T = \lim_i T_i$ of compact operators T_i are compact: given $\varepsilon > 0$, choose T_i so that $|T_i - T|_{\text{op}} < \varepsilon$, and cover the image of the unit ball B_1 under T_i by finitely-many open balls U_k of radius ε. Since $|T_i x - Tx| < \varepsilon$ for all $x \in X$, enlarging the balls U_k to radius 2ε covers TB_1. (If desired, rewrite the proof replacing ε by $\varepsilon/2$.) ///

There exist Banach spaces with compact operators which are *not* norm-limits of finite-rank operators, by [Enflo 1973]. [3]

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[1] A *finite-rank* operator is one with finite-dimensional image.

[2] As usual, the *spectrum* of a continuous operator is the set of $\lambda \in \mathbb{C}$ such that $T - \lambda$ fails to be invertible (to a continuous operator).

[3] Nevertheless, recall that every compact operator $T : X \to Y$ on *Hilbert* spaces *is* an operator-norm limit of finite-rank operators. Indeed, given $\varepsilon > 0$, let $y_1, \ldots, y_n$ be the centers of open ε-balls in Y covering TB_1, where B_1 is the unit ball in X. Let $T_\varepsilon = P \circ T$ where P is the orthogonal projection of Y to the span of the y_i. Then

Similarly, the *sum* of two compact operators is compact.

[1.2] Spectrum of a bounded operator

The *eigenvalues* or *point spectrum* of an operator T on a Banach space X consists of $\lambda \in \mathbb{C}$ such that $T-\lambda$ fails to be *injective*.

The *continuous* spectrum consists of λ with $T-\lambda$ *injective* and with *dense* image, but *not surjective*.

The *residual spectrum* consists of λ with $T-\lambda$ *injective* but $(T-\lambda)X$ not *dense*.

The first serious goal is to prove that all non-zero spectrum of a compact operator on a Banach space is *point spectrum*. This requires some preparation.

[1.3] Compact operators invertible only on finite-dimensional

For compact $T : X \to Y$ with continuous inverse T^{-1}, the boundedness of T^{-1} gives a constant C such that $|T^{-1}y| \le C \cdot |y|$ for all $y \in Y$. Invertibility implies that $TX = Y$, and $|x| \le C \cdot |Tx|$ for all $x \in X$. Thus, the image by T of the unit ball in X contains an open ball in Y. Compactness implies that Y is finite-dimensional, and invertibility implies that X is finite-dimensional.

[1.4] (Generalized) eigenspaces finite-dimensional for $\lambda \neq 0$

For compact $T : X \to X$ and $\lambda \neq 0$, the kernel of $T-\lambda$ is finite-dimensional, since any restriction of T to a subspace is still compact, and T acts by a scalar on $\ker(T-\lambda)$.

By induction on n, the operator $T-\lambda$ maps $\ker(T-\lambda)^{n+1}$ to the finite-dimensional space $\ker(T-\lambda)^n$, so is finite-rank. On $\ker(T-\lambda)^{n-1}$,

$$\text{compact} \;=\; \text{finite-rank} \;=\; T-\lambda \;=\; \text{compact} - \lambda \qquad\qquad (\text{on } \ker(T-\lambda)^{n+1})$$

Thus, $\lambda \neq 0$ is compact on $\ker(T-\lambda)^{n+1}$, implying that this kernel is finite-dimensional. ///

[1.5] Strong duals of Banach spaces

We give the dual X^* of a Banach space its strong topology, from the norm

$$|\lambda|_{X^*} \;=\; \sup_{x \in X,\, |x| \le 1} |Tx|_Y$$

Direct computation shows the dual is *complete*, so is a Banach space.

[1.6] Adjoints

For Banach space maps $T : X \to Y$, the *adjoint* $T^* : Y^* \to X^*$ is characterized by

$$(T^*\eta)(x) \;=\; \eta(Tx) \qquad\qquad (\text{for } x \in X,\, \eta \in Y^*)$$

Continuity is easily verified by checking boundedness:

$$|(T^*\eta)(x)| \;=\; |\eta(Tx)| \;\le\; |\eta|_{Y^*} \cdot |Tx|_Y \;\le\; |\eta|_{Y^*} \cdot |T|_{\rm op} \cdot |x|_X \;\le\; |\eta|_{Y^*} \cdot |T|_{\rm op} \qquad\qquad (\text{for } |x| \le 1)$$

This is mildly incompatible with the Hilbert-space notion, since for Banach spaces $T \to T^*$ is $\mathbb{C}$-linear, while in the Hilbert-space situation the usual convention makes $T \to T^*$ $\mathbb{C}$-*conjugate*-linear.

$|T_\varepsilon - T|_{\rm op} < \varepsilon$, and T_ε is of finite rank.

[1.7] T compact $\iff$ T^* compact

A Banach-space operator T is compact if and only if T^* is compact.

Proof: Take T compact. The closed unit ball $\bar{U}$ in Y^* is *equicontinuous*, as a collection of functions on Y, since

$$|\eta y - \eta y'| \;\le\; |\eta| \cdot |y - y'| \qquad\qquad (\text{for } y, y' \in Y \text{ and } |\eta| \le 1)$$

Since the closure $E = \overline{TB_1}$ of the image TB_1 of the unit ball B_1 of X is *compact*, the restriction of $\bar{U}$ to E is an pointwise-bounded, equicontinuous collection of functions on a compact space. By Arzela-Ascoli, such a collection is a pre-compact subset of $C^o(E)$. Thus, a sequence η_n in $\bar{U}$, restricted to E, has a *uniformly* convergent subsequence η_{n_i}. In other words,

$$|T^*\eta_{n_i} - T^*\eta_{n_j}| \;=\; \sup_{|x|\le 1} |\eta_{n_i}(Tx) - \eta_{n_j}(Tx)| \;\longrightarrow\; 0$$

Thus, $T^*\eta_{n_i}$ is Cauchy in X^*. The latter is complete, so $T^*\eta_{n_i}$ is convergent. Thus, any sequence $T^*\eta_n$ in $T^*\bar{U}$ has a subsequence convergent in X^*, so $T^*\bar{U}$ is pre-compact, and T^* is compact for T compact.

For T^* compact, we have T^{**} compact. Let $i : X \to X^{**}$ be the natural inclusion with X^{**} given its strong (Banach) topology, making i a homeomorphism to its image. Let $j : Y \to Y^{**}$ be the corresponding map for Y, with

$$T^{**} \circ i \;=\; j \circ T \;\;:\;\; X \;\longrightarrow\; Y^{**}$$

The image iB_1 of the closed unit ball of X in X^{**} lies inside the closed unit ball B_1' of X^{**}. Since $T^{**}B_1'$ is totally bounded in Y^{**}, its subset TB_1 is totally bounded in Y^{**}. The image TB_1 is contained in $jY \subset Y^{**}$, so TB_1 is totally bounded in Y. Thus, T is compact. ///

[1.8] $\mathrm{Im}(T - \lambda)$ is closed for $\lambda \neq 0$

For compact T on a Banach space X, for $\lambda \neq 0$, the image $(T - \lambda)X$ is *closed*.

Proof: Let $(T - \lambda)x_n \to y$. First consider the situation that $\{x_n\}$ is *bounded*. Compactness of T yields a convergent subsequence of Tx_n, and we replace x_n by this subsequence. Then $-\lambda x_n = y - Tx_n$ converges to $y - \lim Tx_n$, so x_n is convergent to $x_o \in X$, since $\lambda \neq 0$, and $Tx_o = y$.

To reduce the general case to the previous, first reduce to the case that $T - \lambda$ is *injective*: from above, $\ker(T - \lambda)$ is finite-dimensional, so we can choose a complementary subspace [4] V to $\ker(T - \lambda)$. Since $(T - \lambda)V = (T - \lambda)X$, to prove the image is closed it suffices to consider V, or, equivalently, that $T - \lambda$ is *injective* on X. Since $T - \lambda$ is a continuous bijection to its image, by the *open mapping theorem* it is an isomorphism to its image. Thus, there is $\delta > 0$ such that $|(T - \lambda)x| \ge \delta|x|$.

Returning to the main argument, suppose that $(T - \lambda)x_n \to y_o$. Then $(T - \lambda)(x_m - x_n) \to 0$. By the claim, $x_m - x_n \to 0$, so x_n is *bounded*, reducing to the previous case. ///

[1.9] $T - \lambda$ injective $\Leftrightarrow$ surjective for $\lambda \neq 0$

For T compact on a Banach space X, for $\lambda \neq 0$, injectivity and surjectivity of $T - \lambda$ are equivalent. This is already an interesting case of the *Fredholm alternative.*

Proof: Suppose $T - \lambda$ is injective. Let $V_n = (T - \lambda)^n X$. Since images of Banach spaces under $T - \lambda$ for compact T and $\lambda \neq 0$ are closed, by induction these are *closed* subspaces of X. For $x \notin (T - \lambda)X$ and any

[4] Finite-dimensional subspaces F of a Banach space B have complementary subspaces: choose a basis $x_1, \ldots, x_n$ for F, let $\xi_1, \ldots, \xi_n$ be a dual basis of functionals on F, and extend every ξ_i to a continuous linear functional on B, by *Hahn-Banach.* The simultaneous 0-set of the extensions is a complementary subspace to F.

$y \in X$,

$$(T-\lambda)^n x - (T-\lambda)^{n+1} y \;=\; (T-\lambda)^n \Big(x - (T-\lambda)y \Big)$$

Injectivity of $T-\lambda$ implies that of $(T-\lambda)^n$, so this is not 0. That is, $(T-\lambda)^n x \notin (T-\lambda)^{n+1}X$. Thus, the chain of subspaces V_n is strictly decreasing.

Take $v_n \in V_n$ such that $|v_n| = 1$ and away from V_n, say by

$$\inf_{y \in V_{n+1}} |v_n - y| \;\geq\; \tfrac{1}{2}$$

The effect of T is

$$Tv_m - Tv_{m+n} \;=\; \lambda v_m + (T-\lambda)v_m - Tv_{m+n} \;\in\; \lambda v_m + V_{n+1} \qquad \text{(integers } m \geq 1 \text{ and } n \geq 1)$$

since V_{m+1} is T-stable. Thus,

$$|Tv_m - Tv_{m+n}| \;\geq\; |\lambda| \cdot \tfrac{1}{2}$$

This is impossible, since compact T maps the bounded set $\{v_n\}$ to a pre-compact set. Thus, the chain of subspaces V_n cannot be strictly decreasing, and have surjectivity $(T-\lambda)X = X$.

On the other hand, suppose $T-\lambda$ is *surjective.* Then the adjoint $(T-\lambda)^*$ is *injective.* Since adjoints of compact operators are compact, we already know that $(T-\lambda)^*$ is *surjective.* Then $(T-\lambda)^{**}$ is *injective.* The natural inclusion $X \to X^{**}$ shows that $T-\lambda$ is a *restriction* of $(T-\lambda)^{**}$, so $T-\lambda$ is necessarily injective. ///

[1.10] $\dim \ker(T-\lambda) = \dim \operatorname{coker}(T-\lambda)$ for $\lambda \neq 0$, T compact

This is the *Fredholm alternative* for operators $T-\lambda$ with T compact and $\lambda \neq 0$: *either* $T-\lambda$ is bijective, *or* has non-trivial kernel *and* non-trivial cokernel, of the same dimension.

As above, the compactness of T implies the finite-dimensionality of $\ker(T-\lambda)$ for $\lambda \neq 0$. Dually, for $y_1, \ldots, y_n \in X$ linearly independent modulo $(T-\lambda)X$, by *Hahn-Banach* there are $\eta_1, \ldots, \eta_n \in X^*$ vanishing on the image $(T-\lambda)X$ and $\eta_i(y_j) = \delta_{ij}$. Such η_i are in the kernel of the adjoint $(T-\lambda)^*$. We know T^* is compact, so $\ker(T-\lambda)^*$ is finite-dimensional.

We've proven that injectivity and surjectivity of $T-\lambda$ are equivalent, and that the kernel and cokernel are finite-dimensional. Let $x_1, \ldots, x_m$ (with $m \geq 1$) span the kernel, and let (the images of) $y_1, \ldots, y_n$ (with $n \geq 1$) span the cokernel, and show that $m = n$.

For $m \leq n$, let X' be a closed complementary subspace to the kernel of $T-\lambda$. Let F be the finite-rank operator which is 0 on X' and $Fx_i = y_i$. The adjusted operator $T' = T + F$ is compact. For $(T'-\lambda)x = 0$,

$$(T-\lambda)x \;=\; Fx \;\in\; (T-\lambda)X \cap \operatorname{span} y_1, \ldots, y_n \;=\; \{0\}$$

That is, $T'-\lambda$ is *injective*, so is *surjective*, so $m = n$. In the opposite case $m \geq n$, let $Fx_i = y_i$ for $i \leq n$, and $Fx_i = y_n$ for $i \geq n$. With $T' = T + F$ again, in this case $T'-\lambda$ is *surjective*, so is injective, and $m = n$. ///

[1.11] Discreteness of spectrum of compact operators

For T compact on an infinite-dimensional Banach space, the non-zero spectrum (if any) is *point* spectrum. The number of eigenvalues λ outside a given disk $|\lambda| \leq r$ is *finite* for $r > 0$, and always 0 is in the spectrum.

Proof: For λ not an eigenvalue, we know that $T-\lambda$ is injective *and surjective*, so by the open mapping theorem it is an isomorphism. Thus, indeed, the only non-zero spectrum consists of *eigenvalues.* We also know that eigenspaces are finite-dimensional, for non-zero eigenvalues.

For infinite-dimensional Banach spaces, 0 inevitably lies in the spectrum, otherwise T would be *invertible*. Then $1 = T \circ T^{-1}$ is the composition of a compact operator and a continuous operator, so is continuous, possible only in finite-dimensional spaces.

Suppose there were infinitely-many different eigenvalues $\lambda_1, \lambda_2, \ldots$ outside the closed disk $|\lambda| \le r$ with $r > 0$, with corresponding eigenvectors x_i with $|x_i| = 1$. First, the x_i are linearly independent: let $\sum_i c_i\, x_i = 0$ be a non-trivial linear dependence relation with fewest non-zero c_i's, and apply T: for an index i_o with $c_{i_o} \neq 0$, we obtain a shorter relation by suitable subtraction,

$$0 \;=\; \sum_i \lambda_i\, c_i\, x_i - \lambda_{i_o} \sum_i c_i\, x_i \;=\; \sum_{i \neq i_o} (\lambda_i - \lambda_{i_o})\, c_i\, x_i$$

Thus, there can be no non-trivial linear dependence. With V_n the span of $x_1, x_2, \ldots, x_n$, this implies that the containments $V_n \subset V_{n+1}$ are *strict*. Thus, there exist unit vectors $y_i \in V_i$ with the distance from y_i to V_{i-1} at least $\frac{1}{2}$. Then for $i > j$

$$Ty_i - Ty_j \;=\; \lambda_i y_i + (T - \lambda_i) y_i - Ty_j \;\in\; \lambda_i y_i + V_{i+1}$$

and, thus, $|Ty_i - Ty_j| \ge |\lambda| \cdot \frac{1}{2}$. However, this contradicts the compactness of T. We conclude that there can be only finitely-many eigenvalues larger than $r > 0$. ///

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2. *Appendix: total boundedness and Arzela-Ascoli*

We recall facts about *pre-compactness* and *total-boundedness* in the context of Banach spaces, and corresponding *equicontinuity* and Arzela-Ascoli theorem for Banach space of *functions*.

Pre-compactness and total-boundedness have meaning in not-necessarily-metrizable topological vectors spaces, and in other *uniform spaces*, but we do not need that generality here.

As usual, a set E in a metric space M is *totally bounded* when, given $\varepsilon > 0$, there are finitely-many open ε-balls in M covering E. A set is *pre-compact* when it has compact closure.

[2.0.1] Proposition: A set E in a *complete* metric space M is *pre-compact* if and only if it is *totally bounded*.

Proof: One direction is easy: when E has compact closure $\bar{E}$, given any cover by ε-balls, $\bar{E}$ has a finite subcover, which covers E, so E is totally bounded.

The more serious implication is vaguely reminiscent of the proof of the Baire Category Theorem. Suppose E is totally bounded, and show that any sequence x_i in E has a Cauchy subsequence. Cover E by finitely-many open balls of radius 1. In at least one of these balls there are infinitely-many elements from the sequence. Pick such a ball B_1, and let i_1 be the smallest index so that x_{i_1} lies in this ball.

The set $E \cap B_1$ is still totally bounded, and contains infinitely-many elements from the sequence. Cover it by finitely-many open balls of radius $1/2$, and choose a ball B_2 so that infinitely-many elements of the sequence lie in $E \cap B_1 \cap B_2$. Choose the index i_2 smallest so that both $i_2 > i_1$ and so that x_{i_2} lies inside $E \cap B_1 \cap B_2$.

Inductively obtain indices $i_1 < \ldots < i_n$, and balls B_i of radius $1/i$ with

$$x_i \in E \cap B_1 \cap B_2 \cap \ldots \cap B_i$$

Cover $E \cap B_1 \cap \ldots \cap B_n$ by finitely-many balls of radius $1/(n+1)$ and choose one, B_{n+1}, containing infinitely-many elements of the sequence. Let i_{n+1} be the first index so that $i_{n+1} > i_n$ and so that

$$x_{n+1} \;\in\; E \cap B_1 \cap \ldots \cap B_{n+1}$$

For $m < n$, $d(x_{i_m}, x_{i_n}) \le \frac{1}{m}$, so this subsequence is Cauchy. Completeness of the metric space gives convergence. ///

Recall that a collection E of continuous $\mathbb{C}$-valued functions on a topological space X is *equicontinuous* if, for all $\varepsilon > 0$ and for all $x \in X$, there is a neighborhood N of x such that

$$|f(x') - f(x)| < \varepsilon \qquad \text{(for all } x' \in N\text{, for all } f \in E)$$

[2.0.2] Corollary: (Arzela-Ascoli) Let X be a *compact* Hausdorff topological space, and $C^o(X)$ the Banach space of sup-normed continuous $\mathbb{C}$-valued functions on X. If the functions in an *equicontinuous* subset E of $C^o(X)$ are *uniformly bounded*, then E is a pre-compact subset of $C^o(X)$, and conversely.

Proof: Given the proposition above, we show that uniformly bounded and equicontinuous are equivalent to *totally bounded*.

Suppose the functions in an equicontinuous subset E are uniformly bounded. Let B be a bounded subset of $\mathbb{C}$ in which all the functions in E take their values. Given $\varepsilon > 0$, let U_i be a *finite* cover by open ε-balls. For each $x \in X$, let N_x be a neighborhood of x such that $|f(x') - f(x)| < \varepsilon$ for all $f \in E$ and $x' \in N_x$. By compactness of X, there is a finite subcover N_{x_i}. For each pair x_i, U_j, let $f_{ij} \in C^o(X)$ with $f_{ij}(x_i) \in U_j$, and let

$$V_{ij} = \{f \in C^o(X) : \sup_X |f - f_{ij}| < \varepsilon\}$$

The finitely-many open ε-balls V_{ij} cover E, by design, so E is totally bounded.

On the other hand, suppose E is totally bounded. Given ε, let f_i be the centers of finitely-many open ε-balls in $C^o(X)$ covering E. These finitely-many continuous functions on compact X are uniformly continuous and uniformly bounded. Given $f \in E$, take f_i so that $\sup_X |f - f_i| < \varepsilon$, and observe the natural three-epsilon estimate:

$$|f(x) - f(x')| \le |f(x) - f_i(x)| + |f_i(x) - f_i(x')| + |f_i(x') - f(x')| < 2\varepsilon + |f_i(x) - f_i(x')|$$

Invocation of uniform continuity and boundedness of the finite collection f_i gives the same for E. ///

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