A Good Spectral Theorem

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- Measurable Hilbert bundles
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1. von Neumann algebras, density theorem

The **commutant** of a subring S of a ring R is

$$S' = \{r \in R : rs = sr, \ \ \forall s \in S\}$$

A **von Neumann algebra** is a $*$-stable subalgebra A of the space $\mathcal{B}(V)$ of bounded operators on a Hilbert space V so that A is *closed in the strong topology* on operators.

Theorem: *(von Neumann Density)* Let A be a $*$-stable subalgebra of the space $\mathcal{B}(V)$ of bounded operators on a Hilbert space, and suppose that A contains the identity. Then

$$A'' = \text{ closure of } A \text{ in the strong operator topology}$$

That is, if A is $*$-closed and contains the scalar operators, then the strong-topology closure condition is equivalent to the *purely algebraic* condition

$$A'' = A$$

Very often the von Neumann algebra in question will indeed include scalar operators, but we reserve the right to have it be otherwise.

2. Direct integrals of Hilbert spaces: definitions

Let (Ω, μ) be a measure space. We assume that μ is a positive regular Borel measure. Further, we suppose that Ω is σ-finite, i.e., is a countable union $\bigcup_i \Omega^{(i)}$ of sets $\Omega^{(i)}$ with finite measure. And we suppose that there is a countable collection $\mathcal{M}_o$ of measurable sets in Ω which *generate* the collection of μ-measurable sets, in the sense that for every μ-measurable set $X \subset \Omega$ there is Y in the smallest σ-algebra containing $\mathcal{M}_o$ so that $X - X \cap Y$ and $Y - X \cap Y$ are of measure zero. This is to avoid irrelevant trouble.

To each $s \in \Omega$ assign a Hilbert space V_s with norm $|\,|_s$ and inner product $\langle , \rangle_s$. This assignment is a **Hilbert (space) bundle** $\mathcal{V}$ over **base space** Ω. The **fiber** of the bundle $\mathcal{V}$ at $s \in \Omega$ is the Hilbert space V_s. A **section** x of $\{V_s : s \in \Omega\}$ is a function

$$x : \Omega \to \coprod_{s \in \Omega} V_s$$

so that $x(s) \in V_s$. We *require* also that there be chosen a collection $\mathcal{F}$ of sections so that

- For all $x, y \in \mathcal{F}$ the $\mathbb{C}$-valued function $s \to \langle x(s), y(s) \rangle_s$ is μ-measurable.
- If z is a section of $\{V_s : s \in \Omega\}$ so that for all $x \in \mathcal{F}$ the function $s \to \langle x(s), z(s) \rangle$ is μ-measurable, then $z \in \mathcal{F}$.
- There is a *countable* collection of elements $x_1, x_2, \ldots$ in $\mathcal{F}$ so that for all $s \in \Omega$ the elements $\{x_1(s), x_2(s), \ldots\}$ are *dense* in V_s.

is a choice of **measurable sections** of $\mathcal{V}$. *Choice of $\mathcal{F}$ is part of the specification of a measurable Hilbert bundle.* (There certainly may be many different choices of such collection of measurable sections).

Note that *separability* is quite explicitly required. Sometimes $\{V_s : s \in \Omega\}$ would be called a **measurable field of Hilbert spaces**.

We define the **norm of a measurable section** $x \in \mathcal{F}$ by

$$|x|^2 = \int_\Omega |x(s)|_s^2 \, d\mu(s)$$

If this is zero, then x is a **null section**; if it is finite, then x is a **square-integrable section**. Let

$$\int_\Omega^\oplus V_s \, d\mu(s)$$

denote the collection of square-integrable sections modulo null sections. This is the **direct integral** of the Hilbert spaces V_s, with respect to the measure μ, and with respect to (the choice) $\mathcal{F}$ of measurable sections.

Proposition: The space $\int_\Omega^\oplus V_s \, d\mu(s)$ is a separable Hilbert space.

Proof: This is very similar to the analogous proof for the 'plain' $L^2(X, \mu)$ for an ordinary measure space (X, μ). ♣

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3. Trivializing Hilbert bundles

Here we show that direct integrals of Hilbert spaces are *trivializable* in a sense made precise just below. This fact is essential at certain technical points in the sequel, where we reduce various questions to the much easier analogues for trivial Hilbert bundles.

We first need to be explicit about **morphisms** (i.e., maps) of direct integrals of Hilbert spaces, or of Hilbert bundles, over a fixed base space Ω. Let $\{V_s : s \in \Omega\}$ and $\{V'_s : s \in \Omega\}$ be two such, *with the same base space* Ω, with measurable sections $\mathcal{F}, \mathcal{F}'$, respectively. Let $\{T_s : s \in \Omega\}$ be a collection of continuous linear maps $T_s : V_s \to V'_s$. We evidently must *assume* that for $v \in \mathcal{F}$ the section defined by

$$(Tv)(s) = T_s(v(s))$$

is in $\mathcal{F}'$. Then we say that T **preserves fibers**. If each T_s is an isometry (or at least μ-almost-everywhere is such), then one easily checks that T is an isometry. If T_s is μ-almost-everywhere an isomorphism of Hilbert spaces, then it is easy to show that T is such. And so on.

Similarly, we can define **direct sums** of direct integrals $\{V_s\}, \{V'_s\}$ of Hilbert spaces over Ω: the fiber at s is just $V_s \oplus V'_s$, and the measurable sections are of the form

$$s \to v(s) \oplus v'(s)$$

for $v \in cF$ and $v' \in \mathcal{F}'$. The norm on the direct sum is given by

$$|v \oplus v'|^2 = |v|^2 + |v'|^2$$

where the latter two norms are those in the two original bundles. The norm also has an obvious integral expression.

The simplest of all direct integrals of Hilbert spaces is just $L^2(\Omega, \mu)$ itself, where $V_s = \mathbb{C}$ with the usual inner product. Extending this example somewhat, a more general but still very simple type of direct integral is a direct integral of a **constant** or **trivial Hilbert bundle**, which by definition is one where $V_o = V_s$ does not depend upon $s \in \Omega$. A direct integral of a constant Hilbert bundle may then be viewed as the space $L^2(\Omega, V_o, \mu)$ of *square-integrable* V_o*-valued functions on* Ω, which is defined to be the space of V_o-valued functions f on Ω so that

$$|f| = \left(\int_\Omega \langle f(s), f(s) \rangle_{V_o} \, d\mu(s) \right)^{\frac{1}{2}} < \infty$$

A **countable direct sum** $\hat{\oplus}_i V_i$ of Hilbert spaces V_i (with inner products $\langle , \rangle_i$ is defined to be the collection of sequences $v = \{v_i\}$ with $v_i \in V_i$ with (finite) norm defined by

$$|v|^2 = \sum_i \langle v_i, v_i \rangle$$

Generally, if Ω is a countable *disjoint* union $\Omega = \bigsqcup_i \Omega_i$ of measurable subsets, then there is an obvious isometric isomorphism

$$\int_\Omega^\oplus V_s \, d\mu(s) \approx= \hat{\oplus}_i \int_{\Omega_i}^\oplus V_s \, d\mu(s)$$

For simplicity of statement, we need a convention about cardinalities. In the remainder of this section we will refer to d-dimensional Hilbert spaces. Now, if d is a positive integer, this has the usual meaning, but if $d = \infty$, we mean that the Hilbert space is *separable* but *not finite-dimensional*. Similarly, when we refer to a collection $e_1, \ldots, e_d$ of vectors, if d is a positive integer this has the usual sense, while if $d = \infty$ we mean this notation to refer to the countable collection $e_1, e_2, \ldots$, in which there is no actual element e_∞.

The following proposition effectively asserts that *any* direct integral of Hilbert spaces is isomorphic to a countable direct sum of direct integrals of *trivial* Hilbert bundles, where the sum is indexed only by *dimension*. An isomorphism of a direct integral to a direct integral of a trivial Hilbert bundle is called a **trivialization** of the bundle.

Proposition: Let $\mathcal{V} = \{V_s : s \in \Omega\}$ be a Hilbert bundle with measurable sections $\mathcal{F}$. There are measurable subsets Ω_d with indices $d = 1, 2, \ldots, \infty$ so that for $s \in \Omega_d$ the dimension of V_s is d. For each $d \leq \infty$, there is a d-dimensional Hilbert space $V_o^{(d)}$ so that we have an isometric isomorphism

$$\int_{\Omega_d}^\oplus V_s \, d\mu(s) \approx L^2(\Omega_d, V_o^{(d)}, \mu)$$

which *preserves fibers* and preserves inner products.

Remarks: The proof contains a little more information than the statement, in that a semi-constructive 'procedure' is described by which we obtain the desired isometric isomorphism. There are many possible different version of such a procedure, and it *may* be both more edifying and more comprehensible if the reader simply finds their own version, rather than read the version here.

Proof: Let $v_1, v_2, \ldots$ be a countable collection of sections in $\mathcal{F}$ so that $v_1(s), v_2(s), \ldots$ is dense in the fiber V_s at s, for all $s \in \Omega$. For fixed set I of indices, let $\delta_I(s)$ be the absolute value of the determinant of the matrix of inner products $\langle v_i(s), v_j(s)\rangle_s$ for $i, j \in I$. (The choice of ordering of I doesn't matter since we take absolute value). This function is measurable. The vectors $\{v_i(s) : i \in I\}$ are linearly independent exactly where $\delta_I(s) \neq 0$. Let Δ_d be the supremum of functions δ_I as I varies over all sts of indices with cardinality d. This function is also measurable. Then dim $V_s < d$ if and only if $\Delta_d(s) = 0$.

For $1 \leq d < \infty$, let Ω_d be the measurable set of s where $\Delta_d(s) \neq 0$ but $\Delta_{d+1}(s) = 0$. For $d = \infty$, let Ω_d be the set of s where *no* Δ_d vanishes. These are the desired subsets as in the assertion of the proposition. It is easy to check that the restriction to each Ω_d of $\mathcal{F}$ provides a suitable class of measurable sections for

$$\int_{\Omega_d}^{\oplus} V_s \, d\mu(s)$$

So now suppose that we have a Hilbert bundle all of whose fibers are of the same dimension $d \leq \infty$. We will make a collection $e_1, e_2, \ldots, e_d$ of sections in $\mathcal{F}$ so that for all $s \in \Omega$ the collection $e_1(s), \ldots, e_d(s)$ is an *orthonormal basis* for V_s.

This will give an isomorphism to a trivial bundle, as follows. Let $\mathbb{C}^d$ be the standard d-dimensional Hilbert space, where by convention $d = \infty$ gives the usual space ℓ^2 of square-integrable sequences. For a measurable section v, we have a map to $\mathbb{C}^d$-valued functions on Ω given by

$$v \to (s \to \{\langle v(s), e_i(s)\rangle_s : i = 1, 2, \ldots, d\})$$

It is easy to check that if v is square-integrable then the associated ℓ^d-valued function is square integrable. That is, we have the desired isomorphism to $L^2(\Omega, \mathbb{C}^d, \mu)$.

Again, let $v_1, v_2, \ldots$ be the countable collection whose values pointwise everywhere are dense in the fibers. We will 'improve' v_1 so that $|v_1(s)|_s = 1$ for all $s \in \Omega$, basically by adding to it sections of the form fv_i where $f \in L^\infty(\Omega, \mu)$ and $i > 1$. (By refering to the definition one easily checks that such $fv \in \mathcal{F}$ if $f \in L^\infty(\Omega, \mu)$ and $v \in \mathcal{F}$).

First, we normalize the v_i. Let $f_i(s) = 1$ for $|v_i(s)| = 0$ and $f_i(s) = 1/|v_i|$ for $v_i(s) \neq 0$. Replacing v_i by $f_i v_i$ makes $|v_i(s)|_s \leq 1$ for all s. Since we are only concerned with measurability and not square-integrability at the moment, we do not care about the nature of these functions f_i beyond the fact that they are measurable.

Now let χ_i be the characteristic function of the set of $s \in \Omega$ where $v_1(s) = 0, \ldots, v_{i-1}(s) = 0$ and put

$$v_1' = v_1 + 2^{-1}\chi_2 v_2 + 2^{-2}\chi_3 v_3 + \ldots + 2^{-n}\chi_n v_n + \ldots$$

By construction, the sequence of partial sums is convergent, the closure of the span of $v_1', v_2, \ldots$ is still the whole of V_s at every point, and for all $s \in \Omega$ we have $v_1'(s) \neq 0$. Indeed, for $v_1'(s)$ to vanish requires that *all* the $v_i(s)$ vanish, which is impossible.

And then we may as well replace v_1' by

$$e_1(s) = v_1'(s)/|v_1'(s)|_s$$

to obtain a measurable section e_1 which is pointwise everywhere a unit vector.

Replace the sections $v_2, v_3, ldots$ by the results obtained by applying Gram-Schmidt *pointwise*: replace v_i by

$$v(s) - \langle v_i(s), e_1(s)\rangle_s e_1(s)$$

Then we can repeat the previous process as applied to v_2, replacing it by a section e_2 pointwise everywhere a unit vector orthogonal to e_1. By induction, we use this procedure to construct a sequence e_i of measurable sections so that for every $s \in \Omega$ the vectors $e_i(s)$ are an *orthonormal basis* for the fiber V_s, and in particular so that *none* of these sections vanish at any point $s \in \Omega$.

Note that the procedure *stops* if at any point the Gram-Schmidt process causes all the subsequent sections to become zero. This will be the case if the fibers are all of a fixed finite dimension. Generally, many of the *original* sections v_i will become identically zero along the way, but this does not harm the outcome. ♣

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4. Decomposable operators

For each $s \in \Omega$, let $T(s)$ be a bounded operator on the Hilbert space V_s. If the function

$$s \to \langle T(s)x(s), y(s)\rangle$$

is a measurable $\mathbb{C}$-valued function for all $x, y \in \mathcal{F}$, then say that $\{T_s : s \in \Omega\}$ is a **measurable field** of operators. If the integral converges, then we can form

$$\int_\Omega^\oplus T_s \, d\mu(s)$$

It turns out that we want to require convergence *in the strong topology*, and not in the *uniform* topology.

Some especially simple measurable fields of operators are the **multipliers**: for $f \in L^\infty(\Omega, \mu)$ let M_f be the operator given simply by multiplication by $f(s)$ on V_s. Here of course we have chosen a function representing the *equivalence class* of f in $L^\infty(\Omega, \mu)$, but this does not affect the *multiplier operator*

$$M_f = \int_\Omega^\oplus f(s) \cdot \mathrm{id}_{V_s} \, d\mu(s)$$

obtained on the direct integral space.

Theorem: Let $\mathcal{M}$ be the collection of all multipliers M_f on $V = \int_\Omega^\oplus V_s$ where $f \in L^\infty(\Omega, \mu)$. Let $\mathcal{D}$ be the collection of decomposable operators on V. We have commutant relations

$$\mathcal{M}' = \mathcal{D} \qquad \mathcal{D}' = \mathcal{M}$$

Proof: We can invoke the results above which express a direct integral as a direct sum (indexed by dimension d of fibers) of *trivial* direct integrals

$$L^2(\Omega_d, V, \mu) = \int_{\Omega_d}^\oplus V \, d\mu(s)$$

since the characteristic functions of the subsets Ω_d where the fibers are d-dimensional are *measurable*.

Note that the projectors to the subspaces where the fibers have a fixed dimension are given by multiplication by the characteristic functions of the subsets. That is, these projectors lie in $\mathcal{M}$, and so commute with $T \in \mathcal{M}$. Thus, without loss of generality, we can restrict our attention to the situation that all fibers have the same dimension.

Then, by the previous section, we may suppose that the Hilbert bundle is *trivial*, consisting of the space $L^2(\Omega, V, \mu)$ of square-integrable V-valued functions on Ω (modulo those functions almost everywhere zero, of course), where V is a (separable) Hilbert space.

For $v \in V$ and for *scalar-valued* $\eta \in L^2(\Omega, \mu)$, we use the standard notation

$$\eta \otimes v$$

for the function $s \to \eta(s)v$.

Given an operator $T \in \mathcal{M}'$, define a field $\{T_s\}$ of (fiber-wise) operators by

$$T_s(v) = (T(\eta \otimes v))(s)$$

for $\eta \in L^\infty(\Omega, \mu)$ having compact support and so that $\eta(s) = 1$. Note that we make essential use of the triviality of the bundle in writing such an expression.

We claim that the integral $\int^\oplus T_s$ is none other than T itself.

Proposition: The strong operator topology closure of the collection of multiplier operators M_f with *continuous* f is the collection of multiplier operators M_f with (measurable) *essentially bounded* f.

Proof: Later... ♣

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5. commutative von Neumann Algebras

Theorem: Let $\mathcal{A}$ be a *commutative* von Neumann algebra in $\mathcal{B}(V)$ for a *separable* Hilbert space V. Let A be a C^*-algebra containing 1 and generated by countably-many elements and strong-operator topology dense in $\mathcal{A}$. Let $\Gamma : A \to C^o(\Omega)$ be the Gelfand isomorphism. Then there is a positive regular Borel measure μ (described below) on Ω so that Γ extends to an isomorphism

$$\tilde{\Gamma} : \mathcal{A} \to L^\infty(\Omega, \mu)$$

Lemma: There is $x_o \in V$ so that $\mathcal{A}'x_o$ is dense in V, where $\mathcal{A}'$ is the *commutant* of $\mathcal{A}$. That is, *there is a cyclic vector x_o for the commutant $\mathcal{A}'$.*

Corollary: The functional

$$C^o(\Omega) \ni f \to \langle \Gamma^{-1} f(x_o), x_o \rangle$$

on $C^o(\Omega)$ is *positive*, so gives rise to an outer regular positive Borel measure μ on Ω. Since V is separable, the conclusion of Banach-Alaoglu can be sharpened to see that Ω is *metrizeable*, so μ is necessarily *regular*.

Lemma: Any *other* choice of cyclic vector gives rise to a measure absolutely continuous with respect to μ. Thus, the definition of $\mathrm{L}^\infty(\Omega, \mu)$ is independent of the choice of cyclic vector.

Theorem: Let $\mathcal{A}$ be a commutative von Neumann algebra in $\mathcal{B}(V)$ for a *separable* Hilbert space V. Then there is a Hilbert bundle $\{V_s : s \in \Omega\}$ over a compact metric space Ω so that there is an isometry of Hilbert spaces

$$\Phi : \int_\Omega^\oplus V_s \, d\mu(s) \to V$$

and so that the map

$$L^\infty(\Omega, \mu) \ni f \to \Phi M_f \Phi^{-1}$$

is an *isomorphism.*

Now we describe the Hilbert bundle which occurs in the theorem. The heuristic is that in the prototypical case where Ω is the spectrum of a normal operator T, the fiber V_s is the s-'eigenspace' of T. Of course, this cannot be quite right, since there may be no eigenvalues whatsoever.

For any $x, y \in V$ invoke the Riesz-Markov-Kakutani representation theorem to obtain a regular Borel measure $\mu_{x,y}$ from the functional

$$C^o(\Omega) \ni f \to \langle \Gamma^{-1} f(x), y \rangle$$

A crucial point is that $\mu_{x,y}$ *is absolutely continuous with respect to* $\mu = \mu_{x_o,x_o}$ where x_o is a cyclic vector for $\mathcal{A}'$. Thus,

$$h_{x,y} = \frac{d\mu_{x,y}}{d\mu_{x_o,x_o}} \in L^1(\Omega, \mu)$$

We have the crucial property that for $f \in C^o(\Omega)$ and $x, y \in V$, letting $T = \Gamma^{-1} f \in \mathcal{B}(V)$,

$$h_{Tf(x),y} = f \cdot h_{x,y}$$

Let X be a *countable dense* subset of V, and let Ω_o be a subset of Ω (differing from Ω by a set of measure zero) so that for $s \in \Omega_o$ and $x, y \in X$ the function values $h_{x,y}(s)$ make sense. Let V' be the $\mathbb{C}$-linear span of $\Gamma^{-1} C^o(\Omega) X$ inside V. Then $h_{x,y}(s)$ makes sense for $s \in \Omega_o$ and $x, y \in V'$.

Define

$$(x, y)_s = h_{x,y}(s)$$

This is positive semi-definite on V'. Let K_s be the kernel of this hermitian form. Then $(,)_s$ induces a positive-definite hermitian form denoted $\langle , \rangle_s$ on the quotient V'/K_s. Let V_s be the Hilbert space obtained by completing this space with respect to the induced metric.

For $s \in \Omega_o$ and $x \in X$, let $\tilde{x}(s)$ be the image of x in $V'/K_s \subset V_s$ under the quotient map.

Then $\{V_s : s \in \Omega_o\}$ is a Hilbert bundle over Ω_o with sections $\tilde{x}$ for $x \in X$. Define $\mathcal{F}$ to be the collection of sections y so that

$$s \to \langle \tilde{x}(s), y(s) \rangle$$

is measurable for every $x \in X$. Then

$$V \approx \int_{\Omega_o}^\oplus V_s \, d\mu(s)$$

Since Ω_o differs from Ω by a measure-zero set, the distinction is negligible.

6. Spectral Theorem for a Normal Operator

Let T be a normal operator on a *separable* Hilbert space V. Let A be the C^*-algebra generated by $1, T, T^*$, i.e., the *uniform* operator topology closure of the polynomial ring $\mathbb{C}[T, T^*]$. Let $\mathcal{A}$ be the *strong* operator topology closure of A: this is a von Neumann algebra and is the double commutant A'' of A *and of* $\mathbb{C}[T, T^*]$, by the von Neumann density theorem.

We may identify the maximal ideal space Ω of A with the spectrum $\sigma(T)$ of T. Let Γ be the Gelfand isomorphism

$$\Gamma : A \to C^o(\Omega)$$

From above, we have an extension

$$\tilde{\Gamma} : \mathcal{A} \to L^\infty(\Omega, \mu)$$

where $\mu = \mu_{x_o, x_o}$ is the positive regular Borel measure attached to a cyclic vector x_o for the commutant $\mathcal{A}'$.

For any μ-measurable subset ω of Ω we have the associated $e = e_\omega = \tilde{\Gamma}^{-1}\mathrm{ch}_\omega$ where ch_ω is the characteristic function of ω. If ω is of positive measure, then $e = e_\omega$ is a non-zero idempotent lying in the strong closure A'' of $\mathbb{C}[T, T^*]$.

The function

$$E : \{ \text{ measurable subsets of } \Omega\} \to \{ \text{ self-adjoint projections on } V\}$$

defined by

$$E(\omega) = \tilde{\Gamma}^{-1}(\mathrm{ch}_\omega) \in \mathcal{B}(V)$$

is easily obtained from $\tilde{\Gamma}^{-1}$. This function E is sometimes called a **'resolution of the identity'**.

Note that the properties of the extended Gelfand transform $\tilde{\Gamma}$ immediately yield the required properties of a 'resolution of the identity':

- At the extremes, $E(\emptyset) = 0$ and $E(\Omega) = 1_V$.
- Each $E(\omega)$ is a self-adjoint projection.
- For mesaureable subsets ω, ω', we have
$$E(\omega \cap \omega') = E(\omega) \circ E(\omega')$$
- If $\omega \cap \omega' = \emptyset$ then
$$E(\omega \cup \omega') = E(\omega) + E(\omega')$$
- For all $x, y \in V$ the function
$$E_{x,y}(\omega) := \langle E(\omega)x, y\rangle$$
is a complex measure on Ω.

Indeed, for the last property, note that

$$\langle E(\omega)x, y\rangle = \mu_{x,y}(\omega)$$

in our earlier notation.