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Sobolev imbedding to Lipschitz spaces $C^{k,\alpha}$

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The simplest Sobolev imbedding theorem concerning $\mathbb{R}^n$ asserts that

$$\mathrm{Sob}\big(\frac{n}{2}+k+\varepsilon\big) \;\subset\; C^k \qquad\qquad \text{(for any } \varepsilon > 0)$$

where $C^k = C^k(\mathbb{R}^n)$ is k-times continuously differentiable functions, and the Sobolev space is defined via Fourier transform as

$$\mathrm{Sob}(t) \;=\; \{f \in L^2(\mathbb{R}^n) \;:\; \widehat{f}(x)\cdot(1+|x|^2)^{t/2} \in L^2(\mathbb{R}^n)\}$$

This is easily sharpened to a comparison to Lipschitz spaces

$$C^{k,\alpha}(\mathbb{R}^n) \;=\; \{f \in C^k : |f^{(\beta)}(x) - f^{(\beta)}(y)| \ll |x-y|^\alpha \text{ as } |x-y| \to 0, \text{ for all multi-indices } \beta \text{ with } |\beta| = k\}$$

[0.0.1] Theorem: (Sobolev, *et alia*)

$$\mathrm{Sob}\big(\frac{n}{2}+k+\varepsilon\big) \;\subset\; C^{k,\varepsilon} \qquad\qquad \text{(for any } 0 < \varepsilon < 1)$$

Proof: It suffices to treat $k = 0$, since the Lipschitz condition only applies to the highest-order continuous derivatives. Therefore, we prove

$$\mathrm{Sob}\big(\frac{n}{2}+\varepsilon\big) \;\subset\; C^{0,\varepsilon} \qquad\qquad \text{(for any } 0 < \varepsilon < 1)$$

The inequality defining $C^{0,\varepsilon}$ can be written

$$|\delta_x f - \delta_y f| \;\ll\; |x-y|^\varepsilon$$

For $f \in \mathrm{Sob}(\frac{n}{2}+\varepsilon)$, by the Cauchy-Schwarz-Bunyakowsky inequality for the pairing $\mathrm{Sob}(\frac{n}{2}+\varepsilon)$ and $\mathrm{Sob}(-\frac{n}{2}-\varepsilon)$

$$|\delta_x f - \delta_y f| \;\le\; |f|_{\frac{n}{2}+\varepsilon} \;\cdot\; |\delta_x - \delta_y|_{-\frac{n}{2}-\varepsilon}$$

Thus, it suffices to prove that

$$|\delta_x - \delta_y|_{-\frac{n}{2}-\varepsilon} \;\ll\; |x-y|^\varepsilon$$

To estimate

$$|\delta_x - \delta_y|^2_{-\frac{n}{2}-\varepsilon} \;=\; \int_{\mathbb{R}^n} |e^{i\xi\cdot x} - e^{i\xi\cdot y}|\cdot(1+|\xi|^2)^{-\frac{n}{2}-\varepsilon}\, d\xi$$

use

$$|e^{i\xi\cdot x} - e^{i\xi\cdot y}|^2 \;\ll\; \begin{cases} 1 & (\text{for } |\xi|\cdot|x-y| \ge 1) \\ |\xi|^2\cdot|x-y|^2 & (\text{for } |\xi|\cdot|x-y| \le 1) \end{cases}$$

The integral *outside* the ball is estimated by

$$\int_{|\xi|\ge 1/|x-y|} (1+|\xi|^2)^{-\frac{n}{2}-\varepsilon}\, d\xi \;\le\; \int_{|\xi|\ge 1/|x-y|} |\xi|^{-n-2\varepsilon}\, d\xi \;=\; |x-y|^{2\varepsilon}\int_{|\xi|\ge 1} |\xi|^{-n-2\varepsilon}\, d\xi \;\ll_\varepsilon\; |x-y|^{2\varepsilon}$$

The integral *inside* the ball is estimated by

$$\int_{|\xi|\le 1/|x-y|} |\xi|^2\cdot|x-y|^2\cdot(1+|\xi|^2)^{-\frac{n}{2}-\varepsilon}\, d\xi \;\le\; \int_{|\xi|\le 1/|x-y|} |x-y|^2\cdot|\xi|^{2-n-2\varepsilon}\, d\xi \;=\; |x-y|^{2\varepsilon}\int_{|\xi|\ge 1} |\xi|^{2-n-2\varepsilon}\, d\xi$$

$$\ll\; |x-y|^{2\varepsilon}\int_0^1 r^{2-n-2\varepsilon}\, r^{n-1}\, dr \;\ll\; |x-y|^{2\varepsilon}\int_0^1 r^{1-2\varepsilon}\, dr \;\ll_\varepsilon\; |x-y|^{2\varepsilon}$$

Note that the inequalities succeed only for $0 < \varepsilon < 1$. ///
