(*March 15, 2014*)

Banach and Fréchet spaces of functions

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[This document is http://www.math.umn.edu/˜garrett/m/fun/notes_2012-13/02_spaces_fcns.pdf]

Many familiar and useful spaces of continuous or differentiable functions, such as $C^k[a,b]$, have natural metric structures, and are *complete*. Often, the metric $d(,)$ comes from a *norm* $\|\cdot\|$, on the functions, meaning that

$$d(f,g) \;=\; \|f-g\|$$

where the norm itself has

$$\begin{cases} \|f\| \geq 0, \text{ with } \|f\| = 0 \text{ only for } f = 0 & \text{(positivity)} \\ \|f+g\| \leq \|f\| + \|g\| & \text{(triangle inequality)} \\ \|\alpha \cdot f\| = |\alpha| \cdot \|f\| \;\text{ for } \alpha \in \mathbb{C} & \text{(homogeneity)} \end{cases}$$

A vector space with complete metric coming from a norm is a *Banach space*. Natural Banach spaces of functions are many of the most natural function spaces.

Other natural function spaces, such as $C^\infty[a,b]$ and $C^o(\mathbb{R})$, are *not* Banach, but still have a metric topology and are complete: these are *Fréchet spaces*, appearing as *limits*[1] of Banach spaces. These lack some of the conveniences of Banach spaces, but their expressions as *limits* of Banach spaces is often sufficient.

Other important spaces, such as compactly-supported continuous functions $C_c^o(\mathbb{R})$ on $\mathbb{R}$, or compactly-supported smooth functions $C_c^\infty(\mathbb{R})$ on $\mathbb{R}$, are not reasonably metrizable at all. Some of these important spaces are expressible as *colimits*[2] of Banach or Fréchet spaces, and such descriptions suffice for many applications.

First, we look at some naturally occurring Banach and Fréchet spaces. Our main point will be to prove *completeness* with the natural metrics.

All vector spaces are over the complex numbers $\mathbb{C}$, or possibly over the real numbers $\mathbb{R}$, but usually this will not matter.

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1. *Function spaces* $C^k[a,b]$

Our first examples involve continuous and continuously differentiable functions, $C^o(K)$ and $C^k[a,b]$. The second sort involves measurable functions with integral conditions, the spaces $L^p(X,\mu)$.

In the case of the natural function spaces, the immediate goal is to give the vector space of functions a metric (if possible) which makes it *complete*, so that we can *take limits* and be sure to stay in the same class of functions. For example, *pointwise* limits of continuous functions can easily fail to be continuous.

[1] Examples and ideas about (projective) limits are discussed below.

[2] Examples and ideas about colimits, formerly known as *inductive limits* are discussed carefully below.

[1.0.1] Theorem: The set $C^o(K)$ of (complex-valued) continuous functions on a compact set K is *complete* when given the metric [3]

$$d(f,g) = |f-g|$$

where $|\ |$ is the *norm*

$$|f|_\infty = |f|_{C^o} = \sup_{x\in K} |f(x)|$$

Proof: This is a typical three-epsilon argument. The point is *completeness*, namely that a Cauchy sequence of continuous functions has a *pointwise* limit which is a continuous function. First we observe that a Cauchy sequence f_i has a pointwise limit. Given $\varepsilon > 0$, choose N large enough such that for $i,j \geq N$ we have $|f_i - f_j| < \varepsilon$. Then $|f_i(x) - f_j(x)| < \varepsilon$ for any x in K. Thus, the sequence of values $f_i(x)$ is a Cauchy sequence of complex numbers, so has a limit $f(x)$. Further, given $\varepsilon' > 0$ choose $j \geq N$ sufficiently large such that $|f_j(x) - f(x)| < \varepsilon'$. Then for $i \geq N$

$$|f_i(x) - f(x)| \leq |f_i(x) - f_j(x)| + |f_j(x) - f(x)| < \varepsilon + \varepsilon'$$

Since this is true for every positive ε' we have

$$|f_i(x) - f(x)| \leq \varepsilon$$

for every x in K. (That is, the pointwise limit is approached uniformly in x.)

Now we prove that $f(x)$ is continuous. Given $\varepsilon > 0$, let N be large enough so that for $i,j \geq N$ we have $|f_i - f_j| < \varepsilon$. From the previous paragraph

$$|f_i(x) - f(x)| \leq \varepsilon$$

for every x and for $i \geq N$. Fix $i \geq N$ and $x \in K$, and choose a small enough neigborhood U of x such that $|f_i(x) - f_i(y)| < \varepsilon$ for any y in U. Then

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f(y) - f_i(y)| \leq \varepsilon + |f_i(x) - f_i(y)| + \varepsilon < \varepsilon + \varepsilon + \varepsilon$$

Thus, the pointwise limit f is continuous at every x in U. ///

As usual, a real-valued or complex-valued function f on a closed interval $[a,b] \subset \mathbb{R}$ is **continuously differentiable** if it has a derivative which is itself a continuous function. That is, the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

exists for all $x \in [a,b]$, and the function $f'(x)$ is in $C^o[a,b]$. Let $C^k[a,b]$ be the collection of k-times continuously differentiable functions on $[a,b]$, with the C^k-**norm**

$$|f|_{C^k} = \sum_{0\leq i\leq k} \sup_{x\in[a,b]} |f^{(i)}(x)| = \sum_{0\leq i\leq k} |f^{(i)}|_\infty$$

where $f^{(i)}$ is the i^{th} derivative of f. The **associated metric** on $C^k[a,b]$ is

$$d(f,g) = |f-g|_{C^k}$$

[3] There is no obligation to denote a norm on functions by *double* bars, if context adequately distinguishes a norm on *functions* from the usual norm on *scalars*.

[1.0.2] Theorem: The metric space $C^k[a,b]$ is complete.

Proof: The case $k = 1$ already illustrates the key point. As in the case of C^o just above, for a Cauchy sequence f_n in $C^1[a,b]$ the pointwise limits

$$f(x) = \lim_n f_n(x) \qquad g(x) = \lim_n f'_n(x)$$

exist, are approached uniformly in x, and are continuous functions. We must show that f is continuously differentiable by showing that $g = f'$.

By the fundamental theorem of calculus, for any index i, since f_i is continuous, [4]

$$f_i(x) - f_i(a) = \int_a^x f'_i(t)\,dt$$

By an easy form of the Dominated Convergence Theorem [5]

$$\lim_i \int_a^x f'_i(t)\,dt = \int_a^x \lim_i f'_i(t)\,dt = \int_a^x g(t)\,dt$$

Thus

$$f(x) - f(a) = \int_a^x g(t)\,dt$$

from which $f' = g$. ///

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2. *Function spaces* L^p

The L^p function spaces are perhaps less interesting than the spaces $C^k[a,b]$, but have some technical advantages.

For $1 \le p < \infty$, on a measure space [6] (X,μ) with positive measure μ we have the usual L^p spaces

$$L^p(X,\mu) = \{\text{measurable } f : |f|_p < \infty\} \text{ modulo } \sim$$

with the usual L^p norm

$$|f|_p \;=\; \Big(\int_X |f|^p\,d\mu\Big)^{1/p}$$

and associated metric

$$d(f,g) \;=\; |f-g|_p$$

taking the quotient by the equivalence relation

$$f \sim g \text{ if } f - g = 0 \text{ off a set of measure } 0$$

[4] This invocation of the fundamental theorem of calculus for integrals of *continuous* functions needs only the very simplest notion of an integral.

[5] It may seem to be overkill to invoke the Dominated Convergence Theorem in this context, but attention to such details helps us avoid many of the gaffes of early 19th-century analysis.

[6] The space X on which the measure and the functions live need not be a *topological* space, and the measure μ need have no connection with continuity, for this to make sense.

In the special case that $X = \{1, 2, 3, \ldots\}$ with counting measure μ, the L^p-space is denoted

$$\ell^p \;=\; \{\text{complex sequences } \{c_i\} \text{ with } \Big(\sum_i |c_i|^p\Big)^{1/p} < \infty\}$$

[2.0.1] Remark: These L^p functions have ambiguous pointwise values, in conflict with the naive formal definition of *function*.

[2.0.2] Theorem: The spaces $L^p(X, \mu)$ are complete metric spaces.

[2.0.3] Remark: In fact, as used in the proof, a Cauchy sequence f_i in $L^p(X, \mu)$ has a subsequence converging *pointwise* off a set of measure 0 in X.

Proof: The triangle inequality here is *Minkowski's inequality.* To prove completeness, choose a subsequence f_{n_i} such that

$$|f_{n_i} - f_{n_{i+1}}|_p \;<\; 2^{-i}$$

and put

$$g_n(x) \;=\; \sum_{1\le i\le n} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

and

$$g(x) \;=\; \sum_{1\le i<\infty} |f_{n_{i+1}}(x) - f_{n_i}(x)|$$

The infinite sum is not necessarily claimed to converge to a finite value for every x. The triangle inequality shows that $|g_n|_p \le 1$. *Fatou's Lemma* (itself following from *Lebesgue's Monotone Convergence Theorem*) asserts that for $[0, \infty]$-valued measurable functions h_i

$$\int_X \Big(\liminf_i h_i\Big) \;\le\; \liminf_i \int_X h_i$$

Thus, $|g|_p \le 1$, so is finite. Thus,

$$f_{n_1}(x) + \sum_{i\ge 1} (f_{n_{i+1}}(x) - f_{n_i}(x))$$

converges for almost all $x \in X$. Let $f(x)$ be the sum at points x where the series converges, and on the measure-zero set where the series does not converge put $f(x) = 0$. Certainly

$$f(x) \;=\; \lim_i f_{n_i}(x) \quad \text{(for almost all } x\text{)}$$

Now prove that this almost-everywhere pointwise limit is the L^p-limit of the original sequence. For $\varepsilon > 0$ take N such that $|f_m - f_n|_p < \varepsilon$ for $m, n \ge N$. Fatou's lemma gives

$$\int |f - f_n|^p \le \liminf_i \int |f_{n_i} - f_n|^p \le \varepsilon^p$$

Thus $f - f_n$ is in L^p and hence f is in L^p. And $|f - f_n|_p \to 0$. ///

Of course we are often interested in situations where a measure *does* have a connection with a topology.

[2.0.4] Theorem: For a locally compact Hausdorff topological space X with positive regular Borel measure [7] μ, the space $C^o_c(X)$ of compactly-supported continuous functions is *dense* in $L^p(X, \mu)$.

[7] Recall: a measure is *Borel* if open and closed sets are measurable. A measure is *regular* if the measure of a (measurable) set is the *inf* of the measures of the *open* sets containing it, and is the *sup* of the measures of the *compacts* contained in it.

Proof: From the definition of *integral* attached to a measure, an L^p function is approximable in L^p metric by a *simple* function, that is, a measurable function assuming only finitely-many values. That is, a simple function is a *finite* linear combination of characteristic functions of measurable sets E. Thus, it suffices to approximate characteristic functions of measurable sets by continuous functions. The assumed *regularity* of the measure gives compact K and open U such that $K \subset E \subset U$ and $\mu(U-E) < \varepsilon$, for given $\varepsilon > 0$. Urysohn's lemma says that there is continuous f identically 1 on K and identically 0 off U. Thus, f approximates the characteristic function of E. ///

[2.0.5] Corollary: For locally compact Hausdorff X with regular Borel measure μ, $L^p(X, \mu)$ is the L^p-metric completion of $C^o_c(X)$, the compactly-supported continuous functions. ///

[2.0.6] Remark: *Defining* $L^p(X, \mu)$ to be the L^p completion of $C^o_c(X)$ avoids discussion of ambiguous values on sets of measure zero.

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3. *Normed spaces, Banach spaces*

We define an abstract family of vector spaces with metric topologies, including the standard examples above, namely *normed spaces*. *Complete* normed spaces are *Banach spaces*.

A *normed space* or *pre-Banach* is a vector space V with a non-negative real-valued function $|\,|$ (the *norm*) on it with properties

$$\begin{cases} \textit{(Positivity)}\ |v| \ge 0 \text{ and } |v| = 0 \text{ only for } v = 0 \\ \textit{(Homogeneity)}\ |c \cdot v| = |c| \cdot |v| \text{ for } c \in \mathbb{C} \text{ and } v \in V\text{, with usual complex absolute value } |c| \\ \textit{(Triangle inequality)}\ \text{For } v, w \in V,\ |v + w| \le |v| + |w| \end{cases}$$

For V with norm $|\,|$, there is a natural *metric*

$$d(v, w) \;=\; |v - w|$$

The positivity of the norm assures that $d(v, w) = 0$ implies $v = w$. The homogeneity implies *symmetry* of the metric:

$$d(v, w) = |v - w| = | - (v - w)| = |w - v| = d(w, v)$$

The triangle inequality for the norm implies the *triangle inequality* for the metric:

$$d(x, z) = |x - z| = |(x - y) + (y - z)| \le |x - y| + |y - z| = d(x, y) + d(y, z)$$

Further, because of the way it's defined, such a metric is *translation-invariant* meaning that

$$d(x, y) \;=\; d(x + z, y + z) \qquad \text{(translation invariance)}$$

A normed space V *complete* for the associated metric is a *Banach space*.

Many of the standard examples of naturally normed spaces are complete, though this may require proof. Two important examples already noted are

$$\begin{cases} C^o(X)\text{, with sup norm, is a Banach space, for compact } X \\ C^k[a, b]\text{, with } C^k\text{-norm, is a Banach space, for } -\infty < a < b < +\infty \end{cases}$$

[3.0.1] Remark: When the norm on a Banach space comes from a *positive-definite hermitian inner product* $\langle, \rangle$, meaning that $|f| = \langle f, f\rangle^{\frac{1}{2}}$, the space is called a *Hilbert space*. The inner product gives a very useful geometry that is not available in general Banach spaces. Absence of this geometry in Banach spaces is often a problem. Unfortunately, few natural function spaces are Hilbert space. Nevertheless, for example, the *family* of Banach spaces $C^k[a,b]$ can be systematically *compared* to a *family* of Hilbert spaces, the Levi-Sobolev spaces discussed a bit later.

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4. Non-Banach $C^\infty[a,b]$ as limit of Banach spaces $C^k[a,b]$

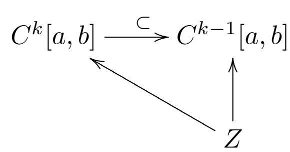
The space $C^\infty[a,b]$ of infinitely differentiable complex-valued functions on a (finite) interval $[a,b]$ in $\mathbb{R}$ is not a Banach space.

Nevertheless, we will see that the topology is *completely determined* by its relation to the Banach spaces $C^k[a,b]$. That is, there is a *unique* reasonable topology on $C^\infty[a,b]$. After proving this uniqueness, we also show that this topology is *complete metric*, although *not* arising from a norm.

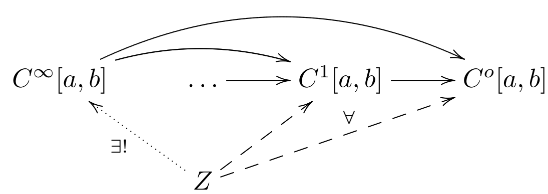
It is useful to observe that this function space can be presented as

$$C^\infty[a,b] \;=\; \bigcap_{k\geq 0} C^k[a,b]$$

A linear map from another vector space Z to $C^\infty[a,b]$ certainly gives a map to each $C^k[a,b]$ by composing with the inclusion $C^\infty[a,b] \to C^k[a,b]$. Conversely, given a family of *continuous linear* maps $Z \to C^k[a,b]$ from a *topological vector space Z compatible* in the sense of giving commutative diagrams



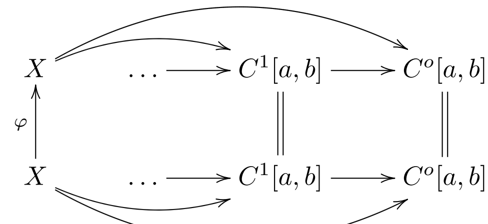
this says that the image of Z actually lies in the intersection $C^\infty[a,b]$. Thus, diagrammatically, for every family of compatible maps $Z \to C^k[a,b]$, there is a *unique* $Z \to C^\infty[a,b]$ fitting into a commutative diagram



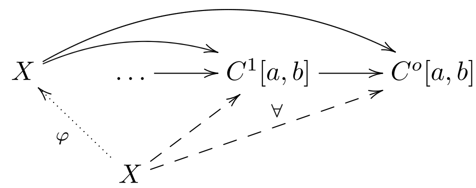
[4.0.1] Theorem: Up to unique isomorphism, there exists at most one topology on $C^\infty[a,b]$ such that to every compatible family of continuous linear maps $Z \to C^k[a,b]$ from a topological vector space Z there is a unique continuous linear $Z \to C^\infty[a,b]$ fitting into a commutative diagram as just above.

[4.0.2] Remark: For the moment, a *topological vector space* is just a (real or complex) vector space with a Hausdorff topology such that vector addition and scalar multiplication are continuous. This includes anything we'll need. But then this uniqueness proof needs amplification for us to see what kind of topology we have.

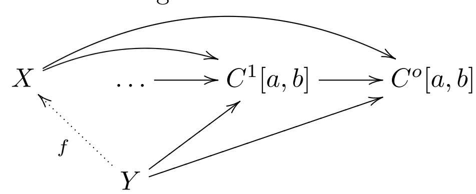
Proof: Let X, Y be $C^\infty[a,b]$ with two topologies fitting into such diagrams, and show $X \approx Y$ (and with unique isomorphism). First, claim that the identity map $\text{id}_X : X \to X$ is the only map $\varphi : X \to X$ fitting into a commutative diagram



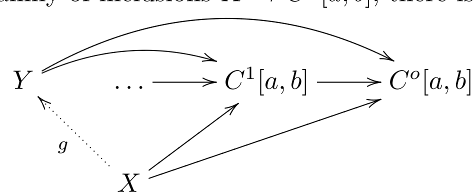
Indeed, given a compatible family of maps $X \to C^k[a,b]$, there is *unique* φ fitting into



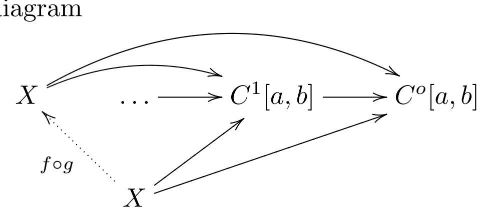
Since the identity map id_X fits, necessarily $\varphi = \mathrm{id}_X$. Similarly, given the compatible family of inclusions $Y \to C^k[a,b]$, there is unique $f : Y \to X$ fitting into



Similarly, given the compatible family of inclusions $X \to C^k[a,b]$, there is unique $g : X \to Y$ fitting into



Then $f \circ g : X \to X$ fits into a diagram



Therefore, $f \circ g = \mathrm{id}_X$. Similarly, $g \circ f = \mathrm{id}_Y$. That is, f, g are mutual inverses, so are isomorphisms of topological vector spaces. ///

Now we prove *existence* of the topology on $C^\infty[a,b]$ by a *construction*. The construction provides some further information, as well.

[4.0.3] Theorem: Let

$$\mu_k(f) \;=\; \sup_{0 \le i \le k} \; \sup_{x \in [a,b]} |f^{(i)}(x)|$$

be the $C^k[a,b]$-norm. With the metric

$$d(f,g) \;=\; \sum_{k=0}^{\infty} 2^{-k} \frac{\mu_k(f-g)}{\mu_k(f-g)+1}$$

$C^\infty[a,b]$ is *complete*.

Proof: A Cauchy sequence f_i in $C^\infty[a,b]$ is also a Cauchy sequence in the Banach space $C^k[a,b]$ for every k. Thus, as we saw earlier, the sequence of restrictions converges in $C^k[a,b]$ to a C^k function on $[a,b]$, for every k, *and* the limit of the derivatives is the corresponding derivative of the limit. ///

[4.0.4] Remark: The particular formula combining all the C^k-metrics can be replaced by many equivalent variants. It is not canonical.

[4.0.5] Remark: The product of any *countable* collection of metric spaces X_i, d_i has a metric given by similar formulas:

$$d(\{x_k\},\{y_k\}) \;=\; \sum_{k=0}^{\infty} 2^{-k} \frac{d_k(x_k,y_k)}{d_k(x_k,y_k)+1}$$

Further, if all the X_i's are *complete*, the product is complete.

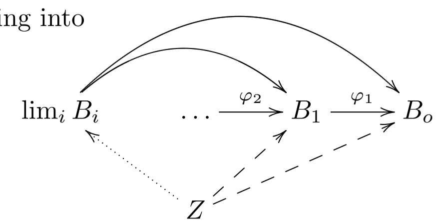
[4.1] Limits of Banach spaces

Abstracting the above, let

$$\ldots \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} B_o$$

be a countable family of Banach spaces with continuous linear maps as indicated. We do *not* require the continuous linear maps to be injective.

A *(projective) limit* $\lim_i B_i$ of the Banach spaces B_i is a topological vector space and continuous linear maps $\lim_i B_i \to B_j$ such that, for every compatible family of continuous linear maps $Z \to B_i$ there is unique continuous linear $Z \to \lim_i B_i$ fitting into



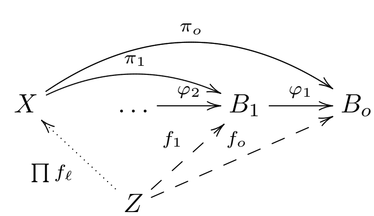
The same *uniqueness* proof as above shows that there is at most one topological vector space $\lim_i B_i$. For *existence* by *construction*, the earlier argument needs minor extension, as follows. The product $P = \prod_i B_i$ of the B_i is complete metrizable, though with no single canonical metric. Let $\pi_i : P \to B_i$ be the i^{th} *projection*. Let

$$X \;=\; \{\{b_i \in B_i\} \in P \;:\; \varphi_\ell(b_\ell) = b_{\ell-1}, \text{ for all } \ell\}$$

be the subset of the product consisting of *compatible* sequences of elements b_i. These are *closed* conditions, so X is a *closed* subset of the product P. By design, the restriction of $\pi_i : P \to B_i$ to $\pi_i : X \to B_i$ satisfies the requirement

$$\varphi_\ell(\pi_\ell\{b_i\}) \;=\; \pi_{\ell-1}(\{b_i\}) \qquad\qquad (\text{for } \{b_i\} \in X)$$

Given any family $f_\ell : Z \to B_\ell$, the product $\prod_\ell f_\ell$ maps $Z \to P$. The condition of *compatibility* on f_ℓ is exactly that $\prod_\ell f_\ell$ has image inside $X \subset P$. In that case, by design, we have a commutative diagram



That is, we have proven *existence* of countable limits of Banach spaces by giving a *construction*.

[4.2] Local convexity

Before declaring $C^\infty[a,b]$ to be a *Fréchet* space, a further technical point must be addressed. A Fréchet space is a topological vector space complete metric with respect to a it translation-invariant metric $d(,)$. The *translation-invariance* means

$$d(f+h, g+h) \;=\; d(f,g)$$

All the metrics above have this property. Further, Fréchet spaces are required to be *locally convex*, that is, at every point there is a local basis of *convex* opens.

For translation-invariant metric, as we will always presume, it suffices to show that 0 has a local basis of convex opens.

Normed spaces are immediately locally convex, because open balls are locally convex, and it suffices: for $0 \le t \le 1$ and x, y in the ε-ball at 0 in a normed space,

$$|tx + (1-t)y| \;\le\; |tx| + |(1-t)y| \;\le\; t|x| + (1-t)|y| \;<\; t\cdot\varepsilon + (1-t)\cdot\varepsilon \;=\; \varepsilon$$

Translation-invariant metrics made as above from countable collections of metrics attached to norms are immediately locally convex. Thus, *countable limits of Banach spaces are locally convex, hence, are Fréchet.*

[4.2.1] Remark: All the spaces we will care about are locally convex for simple reasons, so demonstrating local convexity is rarely an interesting issue. Nevertheless, there are complete-metric topological vectorspaces which *fail* to be locally convex. The sequence space

$$\ell^p \;=\; \{x = (x_1, x_2, \ldots) \;:\; \sum_i |x_i|^p < \infty\}$$

for $0 < p < 1$ with metric

$$d(x,y) \;=\; \sum_i |x_i - y_i|^p \qquad\qquad \text{(note: no } p^{th} \text{ root, unlike the } p \ge 1 \text{ case)}$$

Even this example is of greatest interest merely as a counterexample to a naive presumption that local convexity is automatic.

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5. *Non-Banach function spaces* $C^o(\mathbb{R})$

For a *non-compact* topological space such as $\mathbb{R}$, the space $C^o(\mathbb{R})$ of continuous functions is *not* a Banach space, if for no other reason than that the sup of the absolute value of a continuous function may be $+\infty$.

But, $C^o(X)$ has a complete metric structure under some mild hypotheses on X: suppose that X is a *countable union of compact subsets* K_i, where K_{i+1} contains K_i in its *interior.* [8]

We have *semi-norms*

$$\sup_{x \in K_i} |f(x)|$$

These are called *semi-norms* rather than *norms* since they are not necessarily *strictly positive* for non-zero f, although they are *homogeneous* and satisfy the *triangle inequality.*

The *completion* B_i of $C^o(X)$ with respect to $\sup_{K_i}$ entails *collapsing*, since $\sup_{K_i}$ is only a semi-norm. Nevertheless, the completion of $C^o(X)$ with respect to $\sup_{K_i}$ is a Banach space contained in $C^o(K_i)$. [9]

[8] If we invoke the *Baire Category Theorem*, then we can more simply require that X be locally compact, Hausdorff, and be a countable union of compacts. (The last condition is σ-compactness.) Then it *follows* that, for given $x \in X$, there is some K_i containing a neighborhood of x.

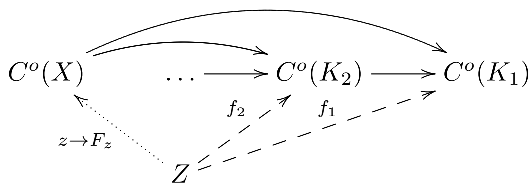
[9] In fact, in tangible examples such as $K_i = [-i,i] \subset \mathbb{R}$, it is easy to see that we have equality $B_i = C^o(K_i)$ rather than mere containment. In the example $K_i = [-i,i] \subset \mathbb{R}$, every continuous function on the compact is a *restriction* of a continuous function on the larger space. However, the argument here does not depend on this.

Despite the fact that the map $C^o(X) \to B_i \subset C^o(K_i)$ need not be injective, we claim that giving $C^o(X)$ the limit topology $\lim_i C^o(K_i)$ is reasonable.

Certainly the restriction map $C^o(X) \to C^{(}K_i)$ should be continuous, as should all the restrictions $C^o(K_i) \to C^o(K_{i-1})$, whether or not these are *surjective*.

Here, the non-formal argument in favor of giving $C^o(X)$ the limit topology is that a *compatible* family of maps $f_i : Z \to C^o(K_i)$ amounts to giving *compatible fragments* of functions F on X. That is, for $z \in Z$, given $x \in X$ take K_i such that x is in the interior of K_i. Then for all $j \geq i$ the function $x \to f_j(z)(x)$ is continuous near x, and the compatibility assures that all these functions are the same.

That is, the compatibility of these fragments is exactly the assertion that they fit together to make a function $x \to F_z(x)$ on the whole space X. Since continuity is a *local* property, $x \to F_z(x)$ is in $C^o(X)$. Further, there is *just one* way to piece the fragments together. Thus, diagrammatically,



Thus, $C^o(X)$ is a Fréchet space.

[5.0.1] Remark: Recall that a non-canonical metric on a countable limit of Banach spaces is given explicitly by

$$d(f,g) \;=\; \sum_i 2^{-i} \frac{\sup_{K_i} |f-g|}{\sup_{K_i} |f-g| + 1}$$

[5.0.2] Remark: When the whole space X is not a *countable* union of compacts, then we cannot form a metric by this procedure. [10] Most physical spaces on which we consider spaces of functions will be σ-compact, if not actually compact.

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6. *Fréchet spaces abstractly*

Non-Banach complete metrizable function spaces, such as $C^\infty[a,b]$ and $C^o(\mathbb{R})$, occur often and are important. In fact, these are examples of countable limit of Banach spaces, which share most properties. With some technical qualifications, these are *Fréchet spaces*, discussed just below.

Let V be a complex vector space with a metric $d(,)$. Suppose d is *translation invariant* in the sense that

$$d(x+z, y+z) \;=\; d(x,y) \qquad\qquad \text{(for all } x,y,z \in V)$$

Note that this property does hold for the metrics induced from norms. Give V the topology induced by the metric. A local basis at $v \in V$ consists of open balls centered at v

$$\{w \in V : d(v,w) < r\}$$

The translation invariance implies that the open balls centered at general points v are the translates

$$v + B_r \;=\; \{v + b : b \in B_r\}$$

[10] One might recall that an *uncountable* sum of positive real numbers cannot converge.

of the open ball B_r of radius r centered at 0. That is, because of the translation invariance, the topology at 0 determines the topology on the whole vector space.

A topology on V is *locally convex* when there is a local basis at 0 (hence, at every point, by translating) consisting of convex sets. A vector space V with a translation-invariant metric $d(,)$ is a *pre-Fréchet space* if the topology is locally convex. When, further, the metric is *complete*, the space V is a *Fréchet space*.

[6.0.1] **Remark**: The local convexity requirement may seem obscure, but does hold in many important cases, such as $C^o(X)$ and $C^k(\mathbb{R})$ treated above, and is crucial for application. But not every metrizable vectorspace is locally convex: ℓ^p with $0 < p < 1$ is the simplest counter-example.

[6.0.2] **Remark**: Later, we will show that every Fréchet space is a countable limit of Banach spaces.