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Fourier series, II

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Banach-Steinhaus/uniform-boundedness has some interesting corollaries for Fourier series.

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1. *Divergence of Fourier series of C^o functions*

The discussion of Levi-Sobolev spaces $H^1(S^1)$ on the circle shows that the Fourier series of periodic C^1 functions f on $\mathbb{R}^1$ converge *uniformly pointwise* (that is, in the C^o metric), and converge to the original f. Thus, in the one dimensional case, the Fourier series of periodic C^1 functions do converge to the functions. However, the Levi-Sobolev approach *suggests* that there should be a difficulty for C^o functions.

It is important that *convergence of Fourier series* means *convergence of the sequence of partial sums.*

The *density* of finite Fourier series in $C^o(S^1)$ makes no claim about *which* finite Fourier series approach a given $f \in C^o(S^1)$. Indeed, the density proof given via the Féjer kernel uses finite Fourier series quite distinct from the finite partial sums of the Fourier series of f itself, namely,

$$N^{th} \text{ Féjer sum} \;=\; \frac{1}{2N+1} \sum_{|n|\le N} (2N+1-|n|) \cdot \widehat{f}(n) \cdot e^{inx}$$

As another failure: the general discussion of L^2 functions shows that a Cauchy sequence of L^2 functions has a *subsequence* converging *pointwise.* Indeed, this proves existence of the limit to prove completeness of L^2. This applies to Fourier series, but does *not* say anything about the pointwise convergence of the *whole* sequence of partial sums, and does not address *uniformity* of the pointwise convergence.

The Banach-Steinhaus theorem for Banach spaces has a decisive corollary about potential convergence failure of Fourier series of $C^o(S^1)$ functions:

[1.0.1] Corollary: There is $f \in C^o(S^1)$ whose Fourier series

$$\sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{inx} \qquad\qquad (\text{with } \widehat{f}(n) = \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} e^{-inx}\, f(x)\, dx)$$

diverges at 0. In fact, the divergence can be arranged at any given countable set of points on S^1.

Proof: To invoke Banach-Steinhaus, consider the functionals given by partial sums of the Fourier series of f, evaluated at 0:

$$\lambda_N(f) \;=\; \sum_{|n|\le N} \hat{f}(n) \;=\; \sum_{|n|\le N} \hat{f}(n) \cdot e^{in\cdot 0}$$

There is an easy upper bound

$$|\lambda_N(f)| \;\le\; \frac{1}{2\pi}\int_0^{2\pi} \Big|\sum_{|n|\le N} e^{-inx}\Big| \cdot \big|f(x)\big|\, dx \;\le\; |f|_{C^o} \cdot \frac{1}{2\pi}\int_0^{2\pi} \Big|\sum_{|n|\le N} e^{-inx}\Big|\, dx \;=\; |f|_{C^o} \cdot \Big|\sum_{|n|\le N} e^{-inx}\Big|_{L^1(S^1)}$$

We will show that equality holds, namely, that

$$|\lambda_N| \;=\; \Big|\sum_{|n|\le N} e^{-inx}\Big|_{L^1}$$

and show that the latter L^1-norms go to ∞ as $N \to \infty$.

Summing the finite geometric series and rearranging:

$$\sum_{|n|\le N} e^{-inx} \;=\; \frac{e^{-iNx} - e^{-i(-N-1)x}}{e^{-ix}-1} \;=\; \frac{e^{i(N+\frac{1}{2})x} - e^{-i(N+\frac{1}{2})x}}{e^{ix}-e^{-ix}} \;=\; \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

The elementary inequality $|\sin t| \le |t|$ gives a lower bound

$$\int_0^{2\pi} \left|\frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}\right| dx \;\ge\; \int_0^{2\pi} \left|\sin(N+\tfrac{1}{2})x\right| \cdot \frac{2}{x}\, dx \;=\; \int_0^{2\pi(N+\frac{1}{2})} |\sin x| \cdot \frac{2}{x}\, dx$$

$$\ge\; \sum_{\ell=1}^{N} \frac{1}{\ell} \int_{2\pi(\ell-1)}^{2\pi\ell} |\sin x|\, dx \;\ge\; \sum_{\ell=1}^{N} \frac{1}{\ell} \;\longrightarrow\; +\infty \qquad (\text{as } N \to \infty)$$

Thus, the L^1-norms do go to ∞.

We claim that the norm of the *functional* is the L^1-norm of the *kernel:* let $g(x)$ be the *sign* of the Dirichlet kernel

$$\sum_{|n|\le N} e^{-inx} \;=\; \frac{\sin(N+\frac{1}{2})x}{\sin\frac{x}{2}}$$

Let g_j be a sequence of periodic continuous functions with $|g_j| \le 1$ and going to g pointwise. [1] By *dominated convergence*

$$\lim_j \lambda_N(g_j) \;=\; \lim_j \int_0^{2\pi} g_j(x) \sum_{|n|\le N} e^{-inx}\, dx = \int_0^{2\pi} g(x) \sum_{|n|\le N} e^{-inx}\, dx = \int_0^{2\pi} |\sum_{|n|\le N} e^{-inx}|\, dx$$

By Banach-Steinhaus for the Banach space $C^o(S^1)$, since (as demonstrated above) there is *no* uniform bound $|\lambda_N| \le M$ for all N, there *exists* f in the unit ball of $C^o(S^1)$ such that

$$\sup_N |\lambda_N\, v| \;=\; +\infty$$

In fact, the collection of such v is *dense* in the unit ball, and is an intersection of a *countable* collection of dense open sets (a G_δ). That is, the Fourier series of f does not converge at 0.

The result can be strengthened by using Baire's theorem again. For a dense countable set of points x_j in the interval, let $\lambda_{j,N}$ be the continuous linear functionals on $C^o(\mathbb{R}/\mathbb{Z})$ defined by evaluation of finite partial sums of the Fourier series at x_j's:

$$\lambda_{j,N}(f) \;=\; \sum_{|n|\le N} \hat{f}(n)\, e^{inx_j}$$

As in the previous argument proof, the set E_j of functions f where

$$\sup_N |\lambda_{j,N} f| \;=\; +\infty$$

is a dense G_δ, so the intersection $E = \bigcap_j E_j$ is a dense G_δ, and, in particular, not empty. ///

[1] The g_j will surely *not* converge *uniformly* pointwise.

2. *Non-surjectivity to* c_o *of* $f \to \widehat{f}$

Recall:

[2.0.1] Theorem: *(Riemann-Lebesgue lemma)* For $f \in L^1[0,1]$

$$\hat{f}(n) \longrightarrow 0$$

Proof: Finite linear combinations of exponentials are dense in $C^o(S^1)$, and $C^o(S^1)$ is dense in $L^1(S^1)$. Thus, given $f \in L^1$ there is $g \in C^o(S^1)$ such that $|f - g|_{L^1} < \varepsilon$ and a finite linear combination h of exponentials such that $|g - h|_{C^o} < \varepsilon$, and then $|f - h|_{L^1} < 2\pi \cdot 2\varepsilon$.

Given such h, for large-enough n the Fourier coefficients are 0, by orthogonality of distinct exponentials. Thus,

$$|\widehat{f}(n)| \;=\; \frac{1}{2\pi}\Big|\int_0^{2\pi} (f(x) - h(x))\, e^{-inx}\, dx\Big| \;\le\; \frac{|f - h|_{L^1}}{2\pi} \;<\; 2\varepsilon \qquad \text{(for } n \text{ large, depending on } f\text{)}$$

This proves the Riemann-Lebesgue Lemma. ///

The space c_o of two-sided sequences *vanishing at infinity* is

$$c_o \;=\; \{\{a_n : n \in \mathbb{Z}\} \;:\; \lim_{|n|\to\infty} a_n = 0\}$$

The space c_o is a Banach space when given norm $|\{a_n\}| \;=\; \sup_n |a_n|$.

[2.0.2] Corollary: *(of Baire and Open Mapping)* It is *not* true that every sequence in c_o is the collection of Fourier coefficients of an $L^1(S^1)$ function.

Proof: The Fourier-coefficient map

$$Tf \;=\; \{\widehat{f}(n) : n \in \mathbb{Z}\} \;\in\; c_0$$

does send $L^1(S^1)$, by the Riemann-Lebesgue lemma. The obvious inequality

$$|\widehat{f}(n)| \;=\; \frac{1}{2\pi}\left|\int_0^{2\pi} f(x)\, e^{-inx}\, dx\right| \;\le\; \frac{1}{2\pi}\int_0^{2\pi} |f(x)|\, dx \;=\; |f|_{L^1}$$

shows $|T| \le 1$. Taking $f(x) = 1$ shows $|T| = 1$.

The density of finite Fourier series in C^o and density of C^o in L^1, as in the proof of the Riemann-Lebesgue lemma, shows that T is *injective.* If T were also *surjective*, then the open mapping theorem would guarantee $\delta > 0$ such that for every L^1 function f

$$|\widehat{f}|_{\sup} \;\ge\; \delta \cdot |f|_{L^1}$$

However, this is impossible: with

$$f_N(x) \;=\; \sum_{|n|\le N} e^{-inx}$$

the sup norm of $\widehat{f}_N$ is certainly 1, yet the computation above shows that the L^1 norm of f_N goes to ∞ as $N \to +\infty$. Thus, there is no such $\delta > 0$. Thus, T cannot be surjective. ///