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Harmonic analysis on compact abelian groups

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The spectral theory for *normal compact* operators on Hilbert spaces, and basic properties of Gelfand-Pettis integrals of vector-valued functions, have immediate application: uniqueness of invariant (Haar) measure on compact abelian groups A, and then proof that

$$L^2(A) \;=\; \text{completion of} \bigoplus_{\chi:A\to\mathbb{C}^\times} \mathbb{C}\cdot\chi$$

where χ runs over continuous *characters* of A, that is, continuous group homomorphisms $A \to \mathbb{C}^\times$. These characters arise as *simultaneous eigenfunctions* for the integral operators

$$T_\varphi \;:\; f \;\longrightarrow\; \int_A \varphi(y)\, f(x+y)\; dy \qquad\qquad (\text{for } \varphi \in C_c^o(A) \text{ and } f \in L^2(A))$$

normalized to $\chi(0) = 1$, writing A additively. This gives another approach to the L^2 theory of Fourier series on circles or products of circles, as well as harmonic analysis on the p-adic integers $\mathbb{Z}_p$, and more exotic items such as *solenoids* $\mathbb{A}/\mathbb{Q}$, where $\mathbb{A}$ is the adele group.

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1. *Approximate identities on topological groups*

[1.1] Topological groups As expected, a *topological group* G is a group with a topology, such that the group operation $G \times G \to G$ and the inversion $G \to G$ are *continuous*. An only-implicit requirement is that G be *locally compact*, and *Hausdorff*. Usually G is required to be *countably based*, to avoid product-measure pathologies. [1]

[1.2] Invariant integrals on topological groups We want an *integral* $f \to \int_G f(g)\, dg$ on $f \in C_c^o(G)$, with the *right invariance*

$$\int_G f(gh)\; dg \;=\; \int_G f(g)\; dg$$

An invariant measure/integral is called a *Haar measure/integral*. For *abelian* G, writing the group operation additively, the invariance condition

$$\int_G f(g+h)\; dg \;=\; \int_G f(g)\; dg$$

[1] Perhaps oddly, this definition of *topological group* excludes infinite-dimensional topological vectorspaces, even though they are (abelian!) *groups* and have *topologies*. However, local compactness or its lack is decisive, so infinite-dimensional topological vectorspaces merit separate treatment. To some degree, the two cases, topological groups and topological vectorspaces, can be subsumed in a common treatment, of *uniform spaces*. Nevertheless, the issue of local compactness or not is pervasive.

is insensitive to left-right issues. [2] We take *existence* of a Haar integral for granted, and prove *uniqueness* below.

[1.3] Continuity of translation action

The *right translation* action of G on any space of functions on G is

$$(R_g f)(x) \;=\; f(xg) \qquad\qquad (\text{for } x, g \in G)$$

The right invariance of the measure/integral immediately gives the invariance of the L^2 norm, for example:

$$|R_g f|^2_{L^2} \;=\; \int_G |f(xg)|^2\, dx \;=\; \int_G |f(x)|^2\, dx \;=\; |f|^2_{L^2}$$

[1.3.1] Claim: The map $G \times L^2(G) \to L^2(G)$ by $g \times f \to R_g f$ is continuous.

Proof: Fix $f \in L^2(G)$, and take $\varepsilon > 0$. Using Urysohn, there is $\varphi \in C^o_c(G)$ such that $|f - \varphi|_{L^2} < \varepsilon$: first approximate f by *simple functions* and then approximate these simple functions by continuous ones, via Urysohn. Since φ is compactly supported, φ is *uniformly* continuous: for all $\varepsilon' > 0$, there is a neighborhood N of $e \in G$ such that $|\varphi(xh) - \varphi(x)| < \varepsilon'$ for all $h \in N$, for all $x \in G$. For $g \in N$,

$$|R_g f - f|_{L^2} \;\le\; |R_g f - R_g \varphi|_{L^2} + |R_g \varphi - \varphi|_{L^2} + |\varphi - f|_{L^2}$$

$$\le\; |f - \varphi|_{L^2} + \varepsilon' \cdot \mathrm{meas}\,(\mathrm{spt}\,\varphi) + |\varphi - f|_{L^2} \;=\; \varepsilon + \varepsilon' \cdot \mathrm{meas}\,(\mathrm{spt}\,\varphi) + \varepsilon$$

Given ε and φ, shrink N so that $\varepsilon' \le \mathrm{meas}\,(\mathrm{spt}\,\varphi)$, so $|R_g f - f|_{L^2} < 3\varepsilon$ for $g \in N$. ///

[1.3.2] Remark: In fact, the crux of the argument is the continuity of the action on $C^o_c(G)$, with its *strict colimit* (LF-space) topology.

[1.4] Approximate identities and Urysohn's lemma

For present purposes, an *approximate identity* $\{\varphi_i\}$ on a topological group G is a sequence of *non-negative* $\varphi_i \in C^o_c(G)$ whose supports shrink to $\{e\}$, where e is the identity in G, in the sense that, given a neighborhood N of e, there is i_o such that for all $i \ge i_o$ the support of φ_i is inside N. Further, given a (right) Haar integral, normalize

$$\int_G \varphi_i(g)\, dg \;=\; 1$$

[1.4.1] Claim: There exists an approximate identity on a topological group G.

Proof: Let N_i be a countable local basis at $e \in G$, ordered so that $N_i \supset N_{i+1}$, and with compact closures. Invoke Urysohn to produce functions ψ_i identically 1 on N_{i+1} and identically 0 off N_i, taking values between 0 and 1. Then normalize $\varphi_i = \psi_i / \int_G \psi_i$. ///

[1.5] Integral-operator action of $C^o_c(G)$ on functions

Let $\varphi \in C^o_c(G)$ act on functions by

$$(\varphi \cdot f)(x) \;=\; \int_G \varphi(g) \cdot f(xg)\, dg$$

[2] For non-abelian groups, it is easy to have a *right*-invariant measure/integral that is not quite *left*-invariant. For example, $G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\}$ with $a \in \mathbb{R}^\times$ and $b \in \mathbb{R}$ has *right*-invariant measure $\frac{da\, db}{|a|}$, but *left*-invariant measure $\frac{da\, db}{|a|^2}$.

Anticipating that continuous, compactly-supported vector-valued Gelfand-Pettis integrals behave well, we can write the action more tersely as

$$\varphi \cdot f \;=\; \int_G \varphi(g) \cdot R_g f \; dg \qquad\qquad \text{(vector-valued integral)}$$

[1.5.1] Claim: For f in a *locally convex* space V of functions on G, and for approximate identity φ_i,

$$\varphi_i \cdot f \;\longrightarrow\; f$$

Proof: Given f and a neighborhood U in V, take a small-enough neighborhood N of e such that $R_g f - f \in U$ for $g \in N$. Take i_o large enough so that for $i \geq i_o$ the support of φ_i is inside N. Then

$$\varphi_i \cdot f - f \;=\; \int_N \varphi_i(g)\, R_g f \; dg - f \;=\; \int_N \varphi_i(g)\, (R_g f - f) \; dg \qquad\qquad (\text{since } \textstyle\int_G \varphi_i = 1)$$

The measure $\varphi_i(g)\, dg$ is a positive, regular Borel measure, with total mass 1. The function $g \to R_g f - f$ on $\operatorname{spt} \varphi_i \subset N$ is a continuous, compactly-supported V-valued function. The fundamental estimate for Gelfand-Pettis integrals is that

$$\int_X F \;\in\; \text{closure of convex hull of } F(X)$$

when X has total measure 1 and F is a continuous, compactly-supported vector-valued function on X. Thus, $\varphi_i \cdot f - f$ is in the closure of the convex hull of all the images $R_g f - f$ for $g \in N$.

Since V is locally convex, without loss of generality U is convex. Further, we can arrange that all the images $R_g f - f$ lie in a smaller convex open U' and $U' + U' \subset U$. Thus, the closure of the convex hull of the images $R_g f - f$ is inside U. ///

[1.5.2] Remark: We will see later that the best hypothesis for V-valued compactly-supported continuous functions to admit Gelfand-Pettis integrals is that V be locally convex and *quasi-complete*.

[1.6] Convolution We do not need to *define* convolution of $C_c^o(G)$ functions, but, rather, *discover* what kind of product on such functions is compatible with repeated application of the integral operators. That is, for $\varphi, \psi \in C_c^o(G)$, we want

$$(\varphi * \psi) \cdot f \;=\; \varphi \cdot \Big(\psi \cdot f\Big)$$

It hardly matters what topological vector space f lies in, whether or not it is a space of functions on G, since the same identity should hold regardless.

Compute directly, using the fact that continuous operators commute with Gelfand-Pettis integrals, and, of course, scalars commute with all linear operators:

$$\varphi \cdot \Big(\psi \cdot f\Big) = \int_G \varphi(g) R_g \int_G \psi(h)\, R_h \; f \; dh\, dg = \int_G \int_G \varphi(g)\psi(h) R_g\, R_h \; f \; dh\, dg = \int_G \int_G \varphi(g)\psi(h) R_{gh} \; f \; dh\, dg$$

At this point, there are two possible courses of action, either replace g by gh^{-1}, or h by $g^{-1}h$. Both choices are completely reasonable, but in the non-commutative case the *appearances* are different. Let's replace g by gh^{-1}, assuming that dg refers to a *right* invariant measure on G:

$$\varphi \cdot \Big(\psi \cdot f\Big) = \int_G \int_G \varphi(gh^{-1})\psi(h) R_g \; f \; dh\, dg = \int_G \Big(\int_G \varphi(gh^{-1})\psi(h) \; dh\Big) R_g f \; dg = \Big(\int_G \varphi(gh^{-1})\psi(h) \; dh\Big) \cdot f$$

That is, we have *proven*

[1.6.1] Proposition: The convolution

$$(\varphi * \psi)(g) \;=\; \int_G \varphi(gh^{-1})\psi(h)\; dh$$

gives the associativity

$$(\varphi * \psi)\cdot f \;=\; \varphi \cdot \Big(\psi \cdot f\Big) \qquad\qquad (\text{for all } f \in L^2(G))$$

This applies to *all* continuous representations of G on reasonable topological vector spaces. ///

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2. *Uniqueness of invariant measure*

Translation-invariant measures on topological groups are *Haar measures*.

We do not prove *existence* of a translation-invariant measure here, but only *uniqueness*.

The space $C_c^o(G)$ is a *strict colimit* of subspaces $C_c^o(E)$ where E ranges over compact subsets of G. Recall that the Riesz-Kakutani-Markov theorem identifies the continuous dual of $C_c^o(G)$ as regular Borel measures.

[2.0.1] Theorem: Let G be a (countably-based, locally compact, Hausdorff) topological group. Then there is a *unique* G-invariant element of the dual space $C_c^o(G)^*$ up to constant multiples, and it is Haar measure

$$f \;\longrightarrow\; \int_G f(g)\, dg$$

[2.0.2] Remark: For simplicity, we assume G is *abelian*, although we write the group operation multiplicatively rather than additively.

Proof: For an approximate identity φ_i in $C_c^o(G)$ and $f \in C_c^o(G)$, we have seen that

$$\varphi_i \cdot f \;=\; \int_G \varphi_i(h)\, R_h f\, dh \;\longrightarrow\; f$$

Let u be an invariant functional on $C_c^o(G)$. By the continuity of u,

$$u(f) \;=\; u\Big(\lim_i g \to \int_G \varphi_i(h)\, f(gh)\, dh\Big) \;=\; \lim_i u\left(g \to \int_G \varphi_i(h)\, f(gh)\, dh\right)$$

$$=\; u\left(g \to \int_G f(h)\, \varphi_i(g^{-1}h)\, dh\right)$$

by replacing h by $g^{-1}h$. By properties of Gelfand-Pettis integrals, and since f and φ_i are compactly-supported continuous functions, the integrand is a compactly-supported V-valued function, and we can move the functional u inside the integral: the above becomes

$$\int_G f(h)\, u\left(g \to \varphi_i(g^{-1}h)\right)\; dh$$

With notation $\check{\varphi}_i(x) = \varphi_i(x^{-1})$, using the abelian-ness of G and the translation-invariance of u, we have

$$u(f) \;=\; \ldots \;=\; \lim_i \int_G f(h)\, u\left(g \to \check{\varphi}_i(h^{-1}g)\right)\; dh \;=\; \lim_i \int_G f(h)\, u\left(g \to \check{\varphi}_i(g)\right)\; dh \;=\; \lim_i u(\check{\varphi}_i)\cdot \int_G f(h)\; dh$$

By assumption the latter expressions approach $u(f)$ as $i \to \infty$. For f so that the latter integral is non-zero, we see that the limit of the $u(\check{\varphi}_i)$ exists, and that $u(f)$ is a constant multiple of the indicated integral with Haar measure. ///

3. *Simultaneous eigenfunctions for integral operators*

Now the abelian-ness and compactness of G will both be used in an essential fashion: the integral operators $f \to \varphi \cdot f$ will form an adjoint-closed commutative ring of Hilbert-Schmidt operators on $L^2(G)$.

[3.1] On compact groups integral operators are Hilbert-Schmidt This is straightforward: for $\varphi \in C_c^o(G)$ and $f \in L^2(G)$,

$$(\varphi \cdot f)(g) \;=\; \int_G \varphi(h)\, f(gh)\; dh \;=\; \int_G \varphi(g^{-1}h)\, f(h)\; dh$$

That is, the operator $f \to \varphi \cdot f$ has *integral kernel* $K(g,h) = \varphi(g^{-1}h)$. Since φ is continuous on a compact space G, K is a continuous function on a finite-measure space $G \times G$, so is in $L^2(G \times G)$, thus giving a Hilbert-Schmidt operator. Thus, these operators are *compact*. The spectral theory of *self-adjoint* compact operators applies to those that are self-adjoint, giving orthogonal bases of corresponding eigenvectors.

[3.2] Integral operators on abelian groups commute This is another direct computation: use the two-sided invariance of the Haar measure, and the invariance of Haar measure under inversion [3] on the group:

$$(\varphi * \psi)(g) \;=\; \int_G \varphi(gh^{-1})\, \psi(h)\; dh \;=\; \int_G \varphi(h^{-1})\, \psi(hg)\; dh \;=\; \int_G \varphi(h)\, \psi(h^{-1}g)\; dh$$

$$=\; \int_G \varphi(h)\, \psi(gh^{-1})\; dh \;=\; (\psi * \varphi)(g)$$

[3.3] Stability under adjoints Let R be the ring of integral operators on $L^2(G)$ containing all operators $T_\varphi : f \to \varphi \cdot f$ for $\varphi \in C_c^o(G)$. We already have

$$T_\varphi \circ T_\psi \;=\; T_{\varphi * \psi}$$

Adjoints are readily determined: for $f, F \in L^2(G)$, successively replace g by $h^{-1}g$, interchange order of integration, and replace h by $gh = hg$, using abelian-ness:

$$\langle T_\varphi f, F\rangle \;=\; \int_G \int_G \varphi(g)\, f(hg)\, \overline{F}(h)\; dg\, dh \;=\; \int_G \int_G \varphi(h^{-1}g)\, f(g)\, \overline{F}(h)\; dg\, dh$$

$$=\; \int_G \int_G \varphi(h^{-1})\, f(g)\, \overline{F}(gh)\; dh\, dg \;=\; \int_G \int_G f(g)\, \overline{\overline{\varphi}(h^{-1})\, F(gh)}\; dh\, dg$$

Thus, letting $\check{\varphi}(h) = \overline{\varphi(h^{-1})}$,

$$T_\varphi^* \;=\; T_{\check{\varphi}}$$

and R is a commutative ring of compact operators closed under adjoints.

[3.4] Simultaneous eigenspaces for commuting operators

The *numerical* notion of *eigenvalue* is insufficient for a *family* of linear operators such as the T_φ.

[3] One way to prove that Haar measure on an abelian group is invariant under inversion is to observe that $f \to \int_G f(g^{-1})\, dg$ is a translation-invariant functional on $C_c^o(G)$, so, by uniqueness of Haar measure, is a multiple of Haar measure...

A commutative ring R of linear operators on a vector space V behaves well with respect to eigenspaces. Namely, given $T \neq 0$ in R and eigenvalue λ for T, every operator $S \in R$ stabilizes the λ^{th} eigenspace V_λ of T: for $v \in V_\lambda$,

$$T(Sv) \;=\; (TS)v \;=\; (ST)v \;=\; S(Tv) \;=\; S(\lambda v) \;=\; \lambda \cdot Sv$$

For $v \neq 0$ a *simultaneous* eigenvector for all operators in R, let $Tv = \mu(T) \cdot v$ for eigenvalue $\mu(T)$. It is immediate that $T \to \mu(T)$ is a *ring homomorphism* $\mu : R \to \mathbb{C}$:

$$\mu(S+T)v \;=\; (S+T)v \;=\; Sv + Tv \;=\; \mu(S)v + \mu(T)v \;=\; \big(\mu(S) + \mu(T)\big)v$$

and

$$\mu(ST)v \;=\; (ST)v \;=\; S(Tv) \;=\; S(\mu(T)v) \;=\; \mu(T) \cdot Sv \;=\; \mu(T)\mu(S)v \;=\; \mu(S)\mu(T)v$$

[3.5] Decomposition by compact operators

[3.5.1] Theorem: A Hilbert space V with an adjoint-closed, commutative $\mathbb{C}$-algebra R of compact operators is the completed direct sum

$$V \;=\; \text{(completion of)} \Big(\bigoplus_{0 \neq \mu : R \to \mathbb{C}} V_\mu \Big) \oplus V_0 \qquad \text{(summed over } \mathbb{C}\text{-algebra homomorphisms } \mu)$$

of simultaneous eigenspaces

$$V_\mu \;=\; \{ v \in V : Tv = \mu(T) \cdot v \text{ for all } T \in R \}$$

For $\mu \neq 0$ the eigenspace V_μ is *finite-dimensional*. The 0-eigenspace may be trivial, finite-dimensional, or infinite-dimensional.

Proof: Note that, since R is adjoint-closed and commutative, every operator $T \in R$ can be written as a linear combination of self-adjoint operators from R:

$$T \;=\; \frac{T + T^*}{2} \;+\; i \cdot \frac{T - T^*}{2i}$$

Let W be the completion of the sum of all simultaneous eigenspaces. Certainly it is R-stable. As usual, the orthogonal complement $W^\perp$ is stable under R: for $w \in W$, $v \in W^\perp$, and $T \in R$,

$$\langle Tv, w \rangle \;=\; \langle v, T^* w \rangle \;=\; 0$$

Suppose W were not all of V. The restrictions of elements of R to W are still compact operators, and the $\mathbb{C}$-algebra of restrictions is closed under adjoints on W. Since W is not contained in the 0-eigenspace of R, there is at least one $T \in R$ with non-zero restriction to W. Without loss of generality, $T = T^*$, and T has a finite-dimensional eigenspace $W_\lambda \subset W$, by the spectral theory for compact self-adjoint operators.

Since it commutes with T, the whole algebra R stabilizes the finite-dimensional W_λ. If there is $T_2 \in R$ whose restriction to W_λ is not a scalar operator, without loss of generality $T_2 = T_2^*$, and there is an eigenspace $\{0\} \neq W_{\lambda_2} \subset W_\lambda$ of T_2 and strictly smaller than W_λ. Continue. Since W_λ is finite-dimensional, a descending chain of subspaces must terminate in finitely-many steps. Thus, there is a non-zero subspace of W_λ which is a simultaneous eigenspace for all R. This contradicts the assumption that W was orthogonal to all simultaneous eigenspaces but non-zero, proving that $W = \{0\}$. ///

[3.6] Triviality of 0-eigenspace

The decomposition by an adjoint-closed commutative ring of compact operators is very general. For the action of $C_c^o(G)$ on $L^2(G)$ for compact abelian G, the decomposition is *non-degenerate*, meaning that there is no 0-eigenspace:

[3.6.1] Corollary: For compact abelian G,

$$L^2(G) \;=\; (\text{completion of}) \bigoplus_{0\neq\mu:R\to\mathbb{C}} L^2(G)_\mu \qquad\qquad (\text{summed over non-zero } \mathbb{C}\text{-algebra homomorphisms } \mu)$$

of simultaneous eigenspaces

$$V_\mu \;=\; \{v \in V : Tv = \mu(T)\cdot v \text{ for all } T \in R\}$$

All eigenspaces V_μ are *finite-dimensional*. The 0-eigenspace is trivial.

Proof: Given $0 \neq f \in L^2(G)$, let φ_i be an approximate identity in $C^o_c(G)$. Since $\varphi_i \cdot f \to f$ in $L^2(G)$, certainly $\varphi_i \cdot f \neq 0$ for large-enough i. Thus, f is not in the simultaneous 0-eigenspace. ///

[3.6.2] Remark: In fact, we can do better: all non-trivial eigenspaces are one-dimensional, as we see in the next section.

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4. *Simultaneous eigenvectors are characters*

For $L^2(G)$ for compact abelian G, a sharper conclusion is possible.

One sense of *character* is a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$. Thus, characters are in $C^o(G)$.

[4.0.1] Corollary: For compact abelian G,

$$L^2(G) \;=\; (\text{completion of}) \bigoplus_{\chi:G\to\mathbb{C}^\times} \mathbb{C}\cdot\chi \qquad\qquad (\text{characters } \chi)$$

The one-dimensional spaces $\mathbb{C}\cdot\chi$ are the simultaneous eigenspaces for the integral-operator action of $C^o_c(G)$.

Proof: That each χ is a simultaneous eigenvector is easy:

$$(\varphi\cdot\chi)(g) \;=\; \int_G \varphi(h)\,\chi(gh)\;dh \;=\; \int_G \varphi(h)\,\chi(g)\,\chi(h)\;dh \;=\; \Big(\int_G \varphi(h)\,\chi(h)\;dh\Big)\cdot\chi(g)$$

It is less obvious that *every* simultaneous eigenvector is of this form. But it is almost immediate that the *translation* operators

$$(T_g f)(h) \;=\; f(hg) \qquad\qquad (\text{for } g,h \in G)$$

commute each other and with the integral operators:

$$(\varphi\circ T_g)f(x) \;=\; \varphi\cdot(x \to f(xg)) \;=\; \int_G \varphi(h)\cdot f(xhg)\;dh \;=\; \int_G \varphi(h)\cdot f(xgh)\;dh$$

$$=\; T_g\Big(x \to \int_G \varphi(h)\cdot f(xh)\;dh\Big) \;=\; (T_g\circ\varphi)f(x)$$

The operators T_g are *unitary*, by changing variables:

$$\langle T_g f, T_g F\rangle \;=\; \int_G f(hg)\,\overline{F}(hg)\;dh \;=\; \int_G f(h)\,\overline{F}(h)\;dh \;=\; \langle f, F\rangle$$

Thus, each of the finite-dimensional $C^o_c(G)$-eigenspaces decomposes into simultaneous eigenspaces for the translation operators. The eigenvalues $\chi : G \to \mathbb{C}^\times$ are group homomorphisms to complex numbers: for eigenfunction f,

$$\chi(gh)f \;=\; T_{gh}f \;=\; T_g(T_h f) \;=\; T_g(\chi(h)\cdot f) \;=\; T_g(\chi(h)\cdot f) \;=\; \chi(h)\cdot T_g f \;=\; \chi(h)\chi(g)f \;=\; \chi(g)\chi(h)f$$

From the unitariness,

$$\chi(g)\overline{\chi}(g)\langle f, f\rangle \;=\; \langle T_g f, T_g f\rangle \;=\; \langle f, f\rangle$$

so $|\chi(g)| = 1$. These characters are *continuous*, by continuity of the translation action: for eigenfunction f,

$$(\chi(g) - \chi(h))f \;=\; T_g f - T_h f \;\longrightarrow\; 0 \qquad\qquad \text{(as } g \to h\text{, by continuity)}$$

For f in the χ-eigenspace,

$$f(g) \;=\; (T_g f)(1) \;=\; \chi(g)\cdot f(1) \;=\; f(1)\cdot\chi(g)$$

That is, f is a scalar multiple of χ, the scalar being $f(1)$. Last, the action of $C_c^o(G)$ does distinguish characters. Indeed, just above we computed that

$$T_\varphi \cdot \chi' \;=\; \Big(\int_G \varphi(h)\,\chi'(h)\,dh\Big)\cdot\chi'$$

In particular,

$$T_{\overline{\chi}} \cdot \chi' \;=\; \Big(\int_G \overline{\chi}(h)\,\chi'(h)\,dh\Big)\cdot\chi'$$

Changing variables in the integral by replacing h by hg, the integral is

$$\int_G \overline{\chi}(h)\,\chi'(h)\,dh \;=\; \overline{\chi}(g)\chi'(g)\cdot\int_G \overline{\chi}(h)\,\chi'(h)\,dh$$

Since $\overline{\chi} = \chi^{-1}$, for $\chi \neq \chi'$ the operator $T_{\overline{\chi}}$ acts by 0 on χ', while $T_{\overline{\chi}}$ acts by a non-zero scalar on χ itself. That is, $C_c^o(G)$ distinguishes characters. That is, each $C_c^o(G)$ contains a single translation-operator eigenspace, which is of the form $\mathbb{C}\cdot\chi$ for a character χ. ///

[4.0.2] Remark: The compact operators reduced to the finite-dimensional situation.
