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Hilbert-Schmidt operators, nuclear spaces, kernel theorem I

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1. Hilbert-Schmidt operators
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Hilbert-Schmidt operators $T : L^2(X) \to L^2(Y)$ are usefully described in terms of their *Schwartz kernels* $K(x, y)$, such that

$$Tf(y) \;=\; \int_Y K(x,y)\, f(x)\, dx$$

Unfortunately, not all continuous linear maps $T : L^2(X) \to L^2(Y)$ have Schwartz kernels, unless one or the other of the two spaces is *finite-dimensional*.

Sufficiently enlarging the class of possible $K(x, y)$ turns out to require a family of topological vector spaces with *tensor products*. [1] The connection between integral/Schwartz kernels and tensor products is suggested by the prototypical Cartan-Eilenberg *adjunction*, for example for k-vectorspaces without topologies: with the usual tensor product of vector spaces,

$$\mathrm{Hom}_k(A, \mathrm{Hom}_k(B, C)) \;\approx\; \mathrm{Hom}(A \otimes_k B, C) \qquad\qquad (\text{by } \varphi \to \big(a \otimes b \to \varphi(a)(b)\big))$$

The special case $C = k$ gives

$$\mathrm{Hom}_k(A, B^*) \;\approx\; \mathrm{Hom}(A \otimes_k B, k) \;=\; (A \otimes_k B)^* \qquad\qquad (k\text{-vectorspaces } A, B, C)$$

That is, maps from A to B^* are given by *integral kernels* in $(A \otimes B)^*$. However, the validity of this adjunction depends on existence of a genuine tensor product. We recall in an appendix the demonstration that infinite-dimensional Hilbert spaces do *not* have tensor products. Also, we must specify the topology on the duals B^* and $(A \otimes B)^*$. The strongest conclusion gives these the *strong* topology, as colimit of Hilbert-space topologies on the duals of Hilbert spaces.

Countable projective limits of Hilbert spaces with transition maps Hilbert-Schmidt constitute the simplest class of *nuclear spaces*: they admit *tensor products*. The simplest example of such a space is the Levi-Sobolev space $H^\infty(\mathbb{T}^n)$ on a product $\mathbb{T}^n$ of circles $\mathbb{T} = S^1$, where the simplest Rellich-Kondrachev compactness lemma is easily refined to prove the requisite Hilbert-Schmidt property.

The main corollary of existence of tensor products of nuclear spaces is *Schwartz' Kernel Theorem*, which provides a framework for later discussion of *pseudo-differential operators*, for example.

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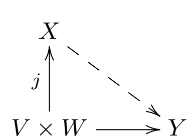
[1] A categorically genuine *tensor product* of topological vector spaces V, W would be a topological vector space X and continuous bilinear map $j : V \times W \longrightarrow X$ such that, for every continuous bilinear $V \times W \longrightarrow Y$ to a topological vector space Y, there is a unique continuous linear $X \longrightarrow Y$ fitting into the commutative diagram



1. *Hilbert-Schmidt operators*

[1.1] Prototype: integral operators

For $K(x,y)$ in $C^o\big([a,b]\times[a,b]\big)$, define $T : L^2[a,b] \to L^2[a,b]$ by

$$Tf(y) \;=\; \int_a^b K(x,y)\, f(x)\, dx$$

The function K is the *integral kernel*, or *Schwartz kernel* of T. Approximating K by *finite* linear combinations of 0-or-1-valued functions shows T is a uniform operator norm limit of finite-rank operators, so is *compact*. The *Hilbert-Schmidt* operators include such operators, where the integral kernel $K(x,y)$ is allowed to be in $L^2\big([a,b]\times[a,b]\big)$.

[1.2] Hilbert-Schmidt norm on $V \otimes_{\text{alg}} W$

In the category of Hilbert spaces and continuous linear maps, there is *no* tensor product in the categorical sense, as demonstrated in an appendix.

Without claiming anything about genuine tensor products in any category of topological vector spaces, the *algebraic* tensor product $X \otimes_{\text{alg}} Y$ of two Hilbert spaces has a hermitian inner product $\langle,\rangle_{\text{HS}}$ determined by

$$\langle x\otimes y, x'\otimes y'\rangle_{\text{HS}} \;=\; \langle x, x'\rangle\,\langle y, y'\rangle$$

Let $X \otimes_{\text{HS}} Y$ be the completion with respect to the corresponding norm $|v|_{\text{HS}} = \langle v, v\rangle_{\text{HS}}^{1/2}$

$$X \otimes_{\text{HS}} Y \;=\; |\cdot|_{\text{HS}}\text{-completion of } X \otimes_{\text{alg}} Y$$

This completion is a Hilbert space.

[1.3] Hilbert-Schmidt operators

For Hilbert spaces V, W the finite-rank [2] continuous linear maps $T : V \to W$ can be identified with the algebraic tensor product $V^* \otimes_{\text{alg}} W$, by [3]

$$(\lambda\otimes w)(v) \;=\; \lambda(v)\cdot w$$

The space of *Hilbert-Schmidt operators* $V \to W$ is the completion of the space $V^* \otimes_{\text{alg}} W$ of finite-rank operators, with respect to the *Hilbert-Schmidt norm* $|\cdot|_{\text{HS}}$ on $V^* \otimes_{\text{alg}} W$. For example,

$$|\lambda\otimes w + \lambda'\otimes w'|^2_{\text{HS}} \;=\; \langle \lambda\otimes w + \lambda'\otimes w', \lambda\otimes w + \lambda'\otimes w'\rangle$$

$$=\; \langle \lambda\otimes w, \lambda\otimes w\rangle + \langle \lambda\otimes w, \lambda'\otimes w'\rangle + \langle \lambda'\otimes w', \lambda\otimes w\rangle + \langle \lambda'\otimes w', \lambda'\otimes w'\rangle$$

$$=\; |\lambda|^2|w|^2 + \langle \lambda, \lambda'\rangle\langle w, w'\rangle + \langle \lambda', \lambda\rangle\langle w', w\rangle + |\lambda'|^2|w'|^2$$

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[2] As usual a *finite-rank* linear map $T : V \to W$ is one with finite-dimensional image.

[3] Proof of this identification: on one hand, a map coming from $V^* \otimes_{\text{alg}} W$ is a *finite* sum $\sum_i \lambda_i \otimes w_i$, so certainly has finite-dimensional image. On the other hand, given $T : V \to W$ with finite-dimensional image, take $v_1, \ldots, v_n$ be an orthonormal basis for the orthogonal complement $(\ker T)^\perp$ of $\ker T$. Define $\lambda_i \in V^*$ by $\lambda_i(v) = \langle v, v_i\rangle$. Then $T \sim \sum_i \lambda_i \otimes Tv_i$ is in $V^* \otimes W$. The second part of the argument uses the completeness of V.

When $\lambda \perp \lambda'$ or $w \perp w'$, the monomials $\lambda \otimes w$ and $\lambda' \otimes w'$ are orthogonal, and

$$|\lambda \otimes w + \lambda' \otimes w'|^2_{\text{HS}} \;=\; |\lambda|^2 |w|^2 + |\lambda'|^2 |w'|^2$$

That is, the space $\text{Hom}_{\text{HS}}(V, W)$ of Hilbert-Schmidt operators $V \to W$ is the *closure* of the space of finite-rank maps $V \to W$, in the space of all continuous linear maps $V \to W$, under the Hilbert-Schmidt norm. By construction, $\text{Hom}_{\text{HS}}(V, W)$ is a Hilbert space.

[1.4] Expressions for Hilbert-Schmidt norm, adjoints

The Hilbert-Schmidt norm of finite-rank $T : V \to W$ can be computed from any choice of orthonormal basis v_i for V, by

$$|T|^2_{\text{HS}} \;=\; \sum_i |Tv_i|^2 \qquad\qquad \text{(at least for finite-rank } T\text{)}$$

Thus, taking a limit, the same formula computes the Hilbert-Schmidt norm of T known to be Hilbert-Schmidt. Similarly, for two Hilbert-Schmidt operators $S, T : V \to W$,

$$\langle S, T\rangle_{\text{HS}} \;=\; \sum_i \langle Sv_i, Tv_i\rangle \qquad\qquad \text{(for any orthonormal basis } v_i\text{)}$$

The Hilbert-Schmidt norm $|\cdot|_{\text{HS}}$ dominates the *uniform operator norm* $|\cdot|_{\text{op}}$: given $\varepsilon > 0$, take $|v_1| \le 1$ with $|Tv_1|^2 + \varepsilon > |T|^2_op$. Choose $v_2, v_3, \ldots$ so that $v_1, v_2, \ldots$ is an orthonormal basis. Then

$$|T|^2_{\text{op}} \;\le\; |Tv_1|^2 + \varepsilon \;\le\; \varepsilon + \sum_n |Tv_n|^2 \;=\; \varepsilon + |T|^2_{\text{HS}}$$

This holds for every $\varepsilon > 0$, so $|T|^2_{\text{op}} \le |T|^2_{\text{HS}}$. Thus, Hilbert-Schmidt limits are operator-norm limits, and Hilbert-Schmidt limits of finite-rank operators are *compact*.

Adjoints $T^* : W \to V$ of Hilbert-Schmidt operators $T : V \to W$ are Hilbert-Schmidt, since for an orthonormal basis w_j of W

$$\sum_i |Tv_i|^2 \;=\; \sum_{ij} |\langle Tv_i, w_j\rangle|^2 \;=\; \sum_{ij} |\langle v_i, T^* w_j\rangle|^2 \;=\; \sum_j |T^* w_j|^2$$

[1.5] Criterion for Hilbert-Schmidt operators

We claim that a continuous linear map $T : V \to W$ with Hilbert space V is Hilbert-Schmidt if for some orthonormal basis v_i of V

$$\sum_i |Tv_i|^2 \;<\; \infty$$

and then (as above) that sum computes $|T|^2_{\text{HS}}$. Indeed, given that inequality, letting $\lambda_i(v) = \langle v, v_i\rangle$, T is Hilbert-Schmidt because it is the Hilbert-Schmidt limit of the finite-rank operators

$$T_n \;=\; \sum_{i=1}^n \lambda_i \otimes Tv_i$$

[1.6] Composition of Hilbert-Schmidt operators with continuous operators

Post-composing: for Hilbert-Schmidt $T : V \to W$ and continuous $S : W \to X$, the composite $S \circ T : V \to X$ is Hilbert-Schmidt, because for an orthonormal basis v_i of V,

$$\sum_i |S \circ Tv_i|^2 \;\le\; \sum_i |S|^2_{\text{op}} \cdot |Tv_i|^2 \;=\; |S|_{\text{op}} \cdot |T|^2_{\text{HS}} \qquad\qquad \text{(with operator norm } |S|_{\text{op}} = \sup_{|v|\le 1} |Sv|\text{)}$$

Pre-composing: for continuous $S : X \to V$ with Hilbert X and orthonormal basis x_j of X, since adjoints of Hilbert-Schmidt are Hilbert-Schmidt,

$$T \circ S \;=\; (S^* \circ T^*)^* \;=\; (\text{Hilbert-Schmidt})^* \;=\; \text{Hilbert-Schmidt}$$

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2. *Simplest nuclear Fréchet spaces*

Roughly, the intention of *nuclear spaces* is that they should admit genuine *tensor products*, aiming at a general Schwartz Kernel Theorem.

For the moment, we consider a more accessible sub-class of nuclear spaces, sufficient for the Schwartz Kernel Theorem for Levi-Sobolev spaces below: countable projective limits of Hilbert spaces with Hilbert-Schmidt transition maps. Thus, they are also Fréchet, so are among *nuclear Fréchet* spaces.

[2.1] $V \otimes_{\text{HS}} W$ is not a categorical tensor product

Again, the Hilbert space $V \otimes_{\text{HS}} W$ is not a categorical tensor product of (infinite-dimensional) Hilbert spaces V, W. In particular, although the bilinear map $V \times W \to V \otimes_{\text{HS}} W$ is continuous, there are (jointly) continuous $\beta : V \times W \to X$ to Hilbert spaces H which do *not* factor through any continuous linear map $B : V \otimes_{\text{HS}} W \to X$.

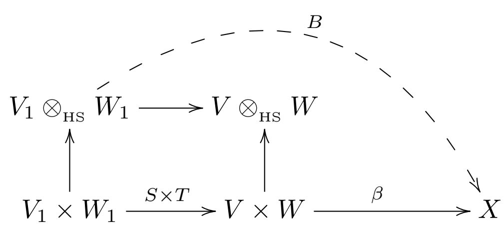
The case $W = V^*$ and $X = \mathbb{C}$, with $\beta(v, \lambda) = \lambda(v)$ already illustrates this point, since not every Hilbert-Schmidt operator has a trace. That is, letting v_i be an orthonormal basis for V and $\lambda_i(v) = \langle v, v_i \rangle$ an orthonormal basis for V^*, necessarily

$$B(\sum_{ij} c_{ij}\, v_i \otimes \lambda_j) \;=\; \sum_{ij} c_{ij} \beta(v_i, \lambda_j) \;=\; \sum_i c_{ii} \qquad\qquad (???)$$

However, $\sum_i \frac{1}{i} v_i \otimes \lambda_i$ is in $V \otimes_{\text{HS}} V^*$, but the alleged value of B is impossible. In effect, the obstacle is that there are Hilbert-Schmidt maps which are not of trace class.

[2.2] Approaching tensor products and nuclear spaces

Let V, W, V_1, W_1 be Hilbert spaces with Hilbert-Schmidt maps $S : V_1 \to V$ and $T : W_1 \to W$. We claim that for any (jointly) continuous $\beta : V \times W \to X$, there is a unique continuous $B : V_1 \otimes_{\text{HS}} W_1 \to X$ giving a commutative diagram



In fact, $B : V_1 \otimes_{\text{HS}} W_1 \to X$ is *Hilbert-Schmidt*. As the diagram suggests, $V \otimes_{\text{HS}} W$ is bypassed, playing no role.

Proof: Once the assertion is formulated, the argument is the only thing it can be: The continuity of β gives a constant C such that $|\beta(v, w)| \le C \cdot |v| \cdot |w|$, for all $v \in V$, $w \in W$. The Hilbert-Schmidt condition is that, for chosen orthonormal bases v_i of V_1 and w_j of W_1,

$$|S|^2_{\text{HS}} \;=\; \sum_i |Sv_i|^2 \;<\; \infty \qquad\qquad |T|^2_{\text{HS}} \;=\; \sum_j |Tw_i|^2 \;<\; \infty$$

Thus,

$$|\beta(Sv, Tw)| \;\le\; C \cdot |Sv| \cdot |Tv|$$

Squaring and summing over v_i and w_j,

$$\sum_{ij} |\beta(Sv_i, Tw_j)|^2 \;\le\; C \cdot \sum_{ij} |Sv_i|^2 \cdot |Tw_j|^2 \;=\; C \cdot |S|^2_{\text{HS}} \cdot |T|^2_{\text{HS}} \;<\; \infty$$

That is, with the obvious definition-attempt

$$B(\sum_{ij} c_{ij}\, v_i \otimes w_j) \;=\; \sum_{ij} c_{ij}\, \beta(Sv_i, Tw_j)$$

Cauchy-Schwarz-Bunyakowsky

$$\sum_{ij} \Big|c_{ij}\, \beta(Sv_i, Tw_j)\Big|^2 \;\le\; \sum_{ij} \Big|c_{ij}\Big|^2 \;\cdot\; \sum_{ij} \Big|\beta(Sv_i, Tw_j)\Big|^2 \;\le\; \sum_{ij} \Big|c_{ij}\Big|^2 \;\cdot\; \Big(C \cdot |S|^2_{\text{HS}} \cdot |T|^2_{\text{HS}}\Big)$$

shows that $B : V_1 \otimes W_1 \to X$ is Hilbert-Schmidt. ///

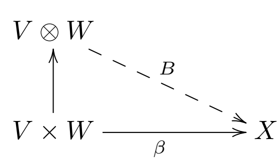
[2.3] A class of nuclear Fréchet spaces

We take the basic *nuclear Fréchet space* to be a countable limit [4] of Hilbert spaces where the transition maps are *Hilbert-Schmidt.*

That is, for a countable collection of Hilbert spaces $V_0, V_1, V_2, \ldots$ with *Hilbert-Schmidt* maps $\varphi_i : V_i \to V_{i-1}$, the limit $V = \lim_i V_i$ in the category of locally convex topological vector spaces is a *nuclear Fréchet space.* [5]

Let $\mathfrak{C}$ be the category of Hilbert spaces enlarged to include limits.

[2.3.1] Theorem: *Nuclear Fréchet spaces admit tensor products* in $\mathfrak{C}$. That is, for nuclear spaces $V = \lim_i V_i$ and $W = \lim W_i$ there is a nuclear space $V \otimes W$ and continuous bilinear $V \times W \to V \otimes W$ such that, given a jointly continuous bilinear map $\beta : V \times W \to X$ of nuclear spaces V, W to $X \in \mathfrak{C}$, there is a unique continuous linear map $B : V \otimes W \to X$ giving a commutative diagram



In particular, $V \otimes W \approx \lim_i V_i \otimes_{\text{HS}} W_i$.

Proof: As will be seen at the end of this proof, the defining property of (projective) limits reduces to the case that X is itself a Hilbert space. Let $\varphi_i : V_i \to V_{i-1}$ and $\psi_i : W_i \to W_{i-1}$ be the transition maps. First, we claim that, for large-enough index i, the bilinear map $\beta : V \times W \to X$ factors through $V_i \times W_i$. Indeed,

[4] Properly, the class of categorical *limits* includes *products* and other objects whose indexing sets are not necessarily *directed.* In that context, requiring that the index set be directed, a projective limit is a *directed* or *filtered* limit. Similarly, what we will call simply *colimits* are properly *filtered* or *directed* colimits.

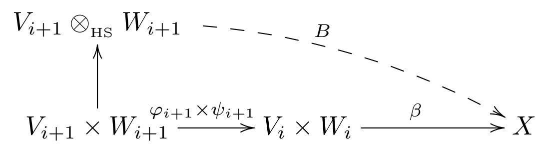
[5] The new aspect is the nuclearity, not the Fréchet-ness: an arbitrary *countable* limit of Hilbert spaces is (provably) Fréchet, since an arbitrary countable limit of *Fréchet* spaces is Fréchet.

the topologies on V and W are such that, given $\varepsilon_o > 0$, there are indices i, j and open neighborhoods of zero $E \subset V_i$, $F \subset W_j$ such that $\beta(E \times F) \subset \varepsilon_o$-ball at 0 in X. Since β is $\mathbb{C}$-bilinear, for *any* $\varepsilon > 0$,

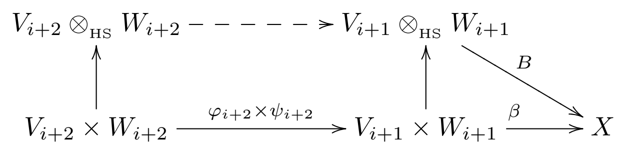
$$\beta(\frac{\varepsilon}{\varepsilon_o} E \times F) \;\subset\; \varepsilon\text{-ball at } 0 \text{ in } X$$

That is, β is already continuous in the $V_i \times W_j$ topology. Replace i, j by their maximum, so $i = j$.

The argument of the previous section exhibits continuous linear B fitting into the diagram



In fact, B is Hilbert-Schmidt. Applying the same argument with X replaced by $V_{i+1} \otimes_{\rm HS} W_{i+1}$ shows that the dotted map in

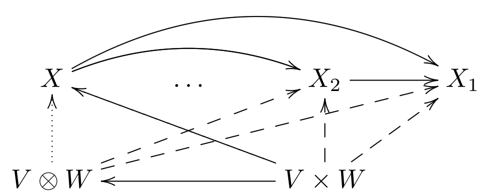


is Hilbert-Schmidt. Thus, the categorical tensor product is the limit of the Hilbert-Schmidt completions of the algebraic tensor products of the limitands:

$$(\lim_i V_i) \otimes (\lim_j W_j) \;=\; \lim_i \big(V_i \otimes_{\rm HS} W_i\big)$$

The transition maps in this limit have been proven Hilbert-Schmidt, so the limit is again nuclear.

As remarked at the beginning of the proof, the general case follows from the basic characterization of projective limits: for $X = \lim_i X_i$ with X_i Hilbert, a continuous bilinear map $V \otimes W \to X$ is exactly a compatible family of maps $V \otimes W \to X_i$. To obtain this compatible family, observe that a continuous bilinear $V \times W \to X$ composed with projections $X \to X_i$ gives a compatible family of continuous bilinear maps $V \times W \to X_i$. These induce compatible linear maps $V \otimes W \to X_i$, as in the commutative diagram



These linear maps $V \otimes W \to X_i$ induce a unique continuous linear $V \otimes W \to X$. ///

[2.4] Example: tensor products of Levi-Sobolev spaces

Let $\mathbb{T}$ be the circle $\mathbb{R}/2\pi\mathbb{Z}$. In terms of Fourier series, for $s \ge 0$ the s^{th} L^2 Levi-Sobolev space on $\mathbb{T}^m$ is

$$H^s(\mathbb{T}^m) \;=\; \{\sum_\xi c_\xi\, e^{i\xi\cdot x} \in L^2(\mathbb{T}^m) \;:\; \sum_\xi |c_\xi|^2 \cdot (1 + |\xi|^2)^s < \infty\}$$

The Levi-Sobolev imbedding theorem asserts that

$$H^{k+\frac{m}{2}+\varepsilon}(\mathbb{T}^m) \;\subset\; C^k(\mathbb{T}^m) \qquad\qquad (\text{for all } \varepsilon > 0)$$

Thus,

$$C^\infty(\mathbb{T}^m) \;=\; H^{+\infty}(\mathbb{T}^m) \;=\; \lim_s H^s(\mathbb{T}^m) \;\approx\; \lim\Big(\ldots \to H^2(\mathbb{T}^m) \to H^1(\mathbb{T}^m) \to H^0(\mathbb{T}^m)\Big)$$

We claim that

$$H^{+\infty}(\mathbb{T}^m) \otimes_{\mathfrak{C}} H^{+\infty}(\mathbb{T}^n) \;\approx\; H^{+\infty}(\mathbb{T}^{m+n})$$

induced from the natural

$$(\varphi \otimes \psi)(x,y) \;=\; \varphi(x)\,\psi(y) \qquad\qquad (\varphi \in H^{+\infty}(\mathbb{T}^m),\, \psi \in H^{+\infty}(\mathbb{T}^n),\, x \in \mathbb{T}^m,\, y \in \mathbb{T}^n)$$

Indeed, our construction of this tensor product is

$$H^{+\infty}(\mathbb{T}^m) \otimes_{\mathfrak{C}} H^{+\infty}(\mathbb{T}^n) \;=\; \lim_s \Big(H^s(\mathbb{T}^m) \otimes_{\rm HS} H^s(\mathbb{T}^n)\Big)$$

The inequalities

$$(1+|\xi|^2+|\eta|^2)^2 \;\geq\; (1+|\xi|^2)(1+|\eta|^2) \;\geq\; 1+|\xi|^2+|\eta|^2 \qquad\qquad (\text{for } \xi \in \mathbb{Z}^m,\, \eta \in \mathbb{Z}^n)$$

give

$$H^{2s}\mathbb{T}^{m+n}) \;\subset\; H^s(\mathbb{T}^m) \otimes_{\rm HS} H^s(\mathbb{T}^n) \;\subset\; H^s(\mathbb{T}^{m+n}) \qquad\qquad (\text{for } s \geq 0)$$

The limit only depends on cofinal sublimits, so, indeed,

$$H^{+\infty}(\mathbb{T}^m) \otimes_{\mathfrak{C}} H^{+\infty}(\mathbb{T}^n) \;\approx\; H^{+\infty}(\mathbb{T}^{m+n})$$

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3. *Schwartz Kernel Theorem for Levi-Sobolev spaces*

Continue the example of Levi-Sobolev spaces on products $\mathbb{T}^m$ of circles $\mathbb{T}$. The following is the simplest example of Schwartz' Kernel Theorem:

[3.0.1] Theorem: We have an *isomorphism*

$$\mathrm{Hom}^o\big(H^\infty(\mathbb{T}^m),\, H^{-\infty}(\mathbb{T}^n)\big) \quad\approx\quad H^{-\infty}(\mathbb{T}^{m+n})$$

induced by

$$\big(f \;\longrightarrow\; (F \to \Phi(f \otimes F)\big) \quad\longleftarrow\quad \Phi \qquad (\text{with } f \in H^\infty(\mathbb{T}^m),\, F \in H^\infty(\mathbb{T}^n),\, \Phi \in H^{-\infty}(T^{m+n}))$$

The distribution $\Phi \in H^{-\infty}(\mathbb{T}^{m+n})$ producing a given continuous map $H^\infty(\mathbb{T}^m) \to H^{-\infty}(\mathbb{T}^n)$ is the *Schwartz kernel* of the map.

[3.0.2] Remark: The Hom-space Hom^o is *continuous* linear maps, so giving sense to the assertion requires a topology on the dual space $H^{-\infty}(\mathbb{T}^n) = H^\infty(\mathbb{T}^n)^*$. The strongest result is true, namely, giving this dual the *strong dual* topology, here meaning the colimit of Hilbert-space topologies on the duals $H^{-s}(\mathbb{T}^n)$ and $H^{-s}(\mathbb{T}^{m+n})$, as opposed to some other topology on those duals of Hilbert spaces. [6]

[6] The *strong* dual topology is traditionally described in other terms, but, later, we show that the traditional and the present sense coincide. There are other useful topologies on duals, such as the *weak* dual topology, which will be seen shortly.

Proof: Let $X = H^\infty(\mathbb{T}^m)$ and $Y = H^\infty(T^n)$, Given the existence of the categorical tensor product, established above, it suffices to show that the vector space

$$\mathrm{Bil}^o(X \times Y, \mathbb{C})$$

of jointly continuous bilinear maps is linearly isomorphic to $\mathrm{Hom}(X, Y^*)$, via the expected

$$\beta \longrightarrow (x \longrightarrow (y \to \beta(x,y))) \qquad \text{(for } \beta \in \mathrm{Bil}^o(X,Y),\ x \in X, \text{ and } y \in Y)$$

where Y^* is given the *strong dual* topology, namely, as colimit of Hilbert-space topologies on the duals $H^{-s}(\mathbb{T}^n)$ with $-s < 0$. The issue is topological.

Given $x \in X$, bounded $E \subset Y$, and $\varepsilon > 0$, by joint continuity of β, there are neighborhoods M, N of 0 in X, Y such that

$$\beta(x + M, N) = \beta(x + M, N) - \beta(x, 0) \subset \varepsilon\text{-ball in } Y^*$$

Since E is bounded, there is $t > 0$ such that $tN \supset E$. Then

$$\beta(x + m, e) - \beta(x, e) = \beta(m, e) \in \beta(M, E) \subset \beta(M, tN) \qquad \text{(for } m \in M \text{ and } e \in E)$$

This suggests replacing M by $t^{-1}M$, so

$$\beta(x + m, e) - \beta(x, e) = \beta(t^{-1}M, E) \subset \beta(t^{-1}M, tN) \subset \varepsilon\text{-ball in } Y^* \qquad \text{(for } m \in t^{-1}M \text{ and } e \in E)$$

That is,

$$\beta(x + m, -) - \beta(x, -) \in U_{E,\varepsilon} \qquad \text{(for } m \in t^{-1}M)$$

This proves the continuity of the map $X \to Y^*$ induced by β.

Conversely, given $\varphi : X \to Y^*$, put $\beta(x, y) = \varphi(x)(y)$. For fixed x, $\beta(x, -) = \varphi(x)$ is continuous, by hypothesis. For fixed y, $E = \{y\}$ is a bounded set in Y, so by the continuity of $x \to \varphi(x)$, for given x and $\varepsilon > 0$ there is a neighborhood M of 0 in X so that $\varphi(x + M) - \varphi(x) \subset U_{E,\varepsilon}$. This proves that $\beta(-, y)$ is continuous. Thus, β is *separately* continuous. An appendix shows that separately continuous bilinear functions on Hilbert spaces are jointly continuous. ///

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4. *Appendix: joint continuity of bilinear maps*

Joint continuity of *separately* continuous bilinear maps on Hilbert spaces, is a corollary of Baire category:

[4.0.1] Claim: A bilinear map $\beta : X \times Y \to Z$ on Hilbert spaces X, Y, Z, continuous in each variable *separately*, is *jointly* continuous.

Proof: Fix a neighborhood N of 0 in Z. Take sequences $x_n \to x_o$ in X and $y_n \to y_o$ in Y. For each $x \in X$, by continuity in Y, $\beta(x, y_n) \to \beta(x, y_o)$. Thus, for each $x \in X$, the set of values $\beta(x, y_n)$ is *bounded* in Z. The linear functionals $x \to \beta(x, y_n)$ are *equicontinuous*, by Banach-Steinhaus, so there is a neighborhood U of 0 in X so that $b_n(U) \subset N$ for all n. In the identity

$$\beta(x_n, y_n) - \beta(x_o, y_o) = \beta(x_n - x_o, y_n) + \beta(x_o, y_n - y_o)$$

we have $x_n - x_o \in U$ for large n, and $\beta(x_n - x_o, y_o) \in N$. Also, by continuity in Y, $\beta(x_o, y_n - y_o) \in N$ for large n. Thus, $\beta(x_n, y_n) - \beta(x_o, y_o) \in N + N$, proving *sequential* continuity. Since $X \times Y$ is metrizable, sequential continuity implies continuity. ///

5. *Appendix: non-existence of tensor products of Hilbert spaces*

Tensor products of infinite-dimensional Hilbert spaces do not exist.

That is, for infinite-dimensional Hilbert spaces V, W, **there is no** Hilbert space X and continuous bilinear map $j : V \times W \longrightarrow X$ such that, for every continuous bilinear $V \times W \longrightarrow Y$ to a Hilbert space Y, there is a unique continuous linear $X \longrightarrow Y$ fitting into the commutative diagram

$$\begin{array}{ccc} X & & \\ {\scriptstyle j}\uparrow & \searrow & \\ V \times W & \longrightarrow & Y \end{array}$$

That is, *there is no tensor product in the category of Hilbert spaces and continuous linear maps.*

Yes, it *is* possible to put an inner product on the *algebraic* tensor product $V \otimes_{\rm alg} W$, by

$$\langle v \otimes w,\, v' \otimes w' \rangle \;=\; \langle v, v' \rangle \cdot \langle w, w' \rangle$$

and extending. The *completion* $V \otimes_{\rm HS} W$, often denoted $V \widehat{\otimes} W$, of $V \otimes_{\rm alg} W$ with respect to the associated norm, is a Hilbert space, identifiable with *Hilbert-Schmidt* operators $V \longrightarrow W^*$. *However*, this Hilbert space fails to have the universal property in the categorical characterization of tensor product, as we see below. This Hilbert space H is important in its own right, but is widely misunderstood as being a tensor product in the categorical sense.

The *non-existence* of tensor products of infinite-dimensional Hilbert spaces is important *in practice*, not only as a cautionary tale [7] about naive category theory, insofar as it leads to Grothendieck's idea of *nuclear spaces*, which *do* admit tensor products.

Proof: First, we review the point that the Hilbert-Schmidt tensor product $H = V \widehat{\otimes} W$ is *not* a Hilbert-space tensor product. For simplicity, suppose that V, W are *separable*, in the sense of having countable Hilbert-space bases.

Choice of such bases allows an identification of W with the continuous linear Hilbert space dual V^* of V. Then we have the continuous bilinear map $V \times V^* \longrightarrow \mathbb{C}$ by $v \times \lambda \longrightarrow \lambda(v)$. The *algebraic* tensor product $V \otimes_{\rm alg} W$ injects to $H = V \widehat{\otimes} V^*$, and the image is identifiable with the *finite-rank* maps $V \longrightarrow V$. The linear map $T : H \longrightarrow \mathbb{C}$ induced on the image of $V \otimes_{\rm alg} V^*$ is *trace*. If $H = V \widehat{\otimes} V^*$ were a Hilbert-space tensor product, the trace map would extend continuously to it from finite-rank operators. However, there are many Hilbert-Schmidt operators that are not of trace class. For example, letting e_i be an orthonormal basis, the element

$$\sum_n \frac{1}{n} \cdot e_n \otimes e_n \quad \in \quad V \widehat{\otimes} V^*$$

does not have a finite trace, since $\sum_{n \le N} 1/n \sim \log N$. In other words, the difficulty is that

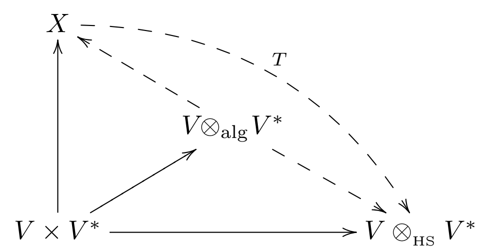
$$T\Big(\sum_{a \le n \le b} \frac{1}{n} \cdot e_n \otimes e_n \Big) \;=\; \sum_{a \le n \le b} \frac{1}{n} \cdot T(e_n \otimes e_n) \;=\; \sum_{a \le n \le b} \frac{1}{n}$$

[7] Many of us are not accustomed to worry about *existence* of objects defined by universal mapping properties, because we proved their existence by set-theoretic *constructions* of them, long before becoming aware of mapping-property characterizations. Much as naive set theory does not lead to paradoxes without effort, naive category theory's recharacterization of objects close to prior experience rarely describes non-existent objects. Nevertheless, the present example is genuine.

Thus, the partial sums of $\sum_n \frac{1}{n}\, e_n \otimes e_n$ form a Cauchy sequence, but the values of T on the partial sums go to $+\infty$. Thus, the Hilbert-Schmidt tensor product cannot be a Hilbert-space tensor product.

Now we show that no *other* Hilbert space can be a tensor product, by comparing to the Hilbert-Schmidt tensor product.

Let $V \times W \longrightarrow X$ be a purported Hilbert-space tensor product, and, again, let W be the dual of V, without loss of generality. By assumption, the continuous bilinear injection $V \times V^* \longrightarrow V \otimes_{\text{HS}} V^*$ induces a unique continuous linear map $T : X \longrightarrow H$ fitting into a commutative diagram



The linear map $V \otimes_{\text{alg}} V^* \longrightarrow V \otimes_{\text{HS}} V^*$ is injective, since $V \otimes_{\text{HS}} V^*$ is a completion of $V \otimes_{\text{alg}} V^*$. Thus, unsurprisingly, $V \otimes_{\text{alg}} V^* \longrightarrow X$ is necessarily injective. The uniqueness of the linear induced maps implies that the image of $V \otimes_{\text{alg}} V^*$ is *dense* in X. Also, $T : X \longrightarrow V \otimes_{\text{HS}} V^*$ is the identity on the copies of $V \otimes_{\text{alg}} V^*$ imbedded in X and $V \otimes_{\text{HS}} V^*$. Let $T^* : V \otimes_{\text{HS}} V^* \longrightarrow X$ be the adjoint of T, defined by

$$\langle x, T^* y\rangle_X \;=\; \langle Tx, y\rangle_{V \otimes_{\text{HS}} V^*}$$

On the imbedded copies of $V \otimes_{\text{alg}} V^*$

$$\langle v \otimes \lambda, T^*(w \otimes \mu)\rangle_X \;=\; \langle T(v \otimes \lambda), w \otimes \mu\rangle_{V \otimes_{\text{HS}} V^*} \;=\; \langle v \otimes \lambda, w \otimes \mu\rangle_{V \otimes_{\text{HS}} V^*} \qquad (\text{for } v, w \in V \text{ and } \lambda, \mu \in V^*)$$

Given $v \in V$ and $\lambda \in V^*$, the orthogonal complement $(v \otimes \lambda)^\perp$ is the closure of the span of monomials $v' \otimes \lambda'$ where *either* $v' \perp v$ *or* $\lambda' \perp \lambda$. For such $v' \otimes \lambda'$,

$$0 \;=\; \langle v' \otimes \lambda', v \otimes \lambda\rangle_H \;=\; \langle T(v' \otimes \lambda'), v \otimes \lambda\rangle_H \;=\; \langle v' \otimes \lambda', T^*(v \otimes \lambda)\rangle_X$$

Thus, for any monomial $v \otimes \lambda$, the image $T^*(v \otimes \lambda)$ is a scalar multiple of $v \otimes \lambda$. The same is true of monomials $(v + w) \otimes (\lambda + \mu)$. Taking v, w linearly independent and λ, μ linearly independent and expanding shows that the scalars do not depend on v, λ. Thus, T^* is a scalar on $V \otimes_{\text{alg}} V^*$.

That is, there is a (necessarily real) constant C such that

$$C \cdot \langle v \otimes \lambda, w \otimes \mu\rangle_X \;=\; \langle v \otimes \lambda, T^*(w \otimes \mu)\rangle_X \;=\; \langle T(v \otimes \lambda), w \otimes \mu\rangle_{V \otimes_{\text{HS}} V^*} \;=\; \langle v \otimes \lambda, w \otimes \mu\rangle_{V \otimes_{\text{HS}} V^*}$$

since T identifies the imbedded copies of $V \otimes_{\text{alg}} V^*$. That is, up to the constant C, the inner products from X and $V \otimes_{\text{HS}} V^*$ restrict to the same hermitian form on $V \otimes_{\text{alg}} V^*$. Thus, any putative tensor product X differs from $V \otimes_{\text{HS}} V^*$ only by scaling. However, we saw that the natural pairing $V \times V^* \longrightarrow \mathbb{C}$ does not factor through a continuous linear map $V \otimes_{\text{HS}} V^* \longrightarrow \mathbb{C}$, because there exist Hilbert-Schmidt maps not of trace class.

Thus, there is no tensor product of infinite-dimensional Hilbert spaces. ///

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