(*April 20, 2014*)

Topological vectorspaces

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/fun/notes_2012-13/07a_general_tvs.pdf]

This is the first introduction to *topological vectorspace* in general. This is useful *after* acquaintance with Hilbert spaces, Banach spaces, Fréchet spaces, to understand important examples *outside* these classes of spaces.

Basic concepts make sense *without* a metric. Some concepts *appearing* to depend a metric have a useful sense in a general context.

Even in this generality, *finite-dimensional* topological vectorspaces have just one reasonable topology. This has immediate consequences for maps to and from finite-dimensional topological vectorspaces.

All this works with mild hypotheses on the scalars involved, although our primary interest is complex scalars, and occasionally real scalars.

---

1. *Natural non-Fréchet spaces*

There are many natural spaces of *functions* that are *not* Fréchet spaces. For example, let

$$C^o_c(\mathbb{R}) \;=\; \{\text{compactly-supported continuous } \mathbb{C}\text{-valued functions on } \mathbb{R}\}$$

This is a strictly smaller space than the space $C^o(\mathbb{R})$ of *all* continuous functions on $\mathbb{R}$, which we saw *is* Fréchet, its topology being given by a countable collection of seminorms. The function space $C^o_c(\mathbb{R})$ is an *ascending union*

$$C^o_c(\mathbb{R}) \;=\; \bigcup_{N=1}^{\infty} \{f \in C^o_c(\mathbb{R}) : \mathrm{spt} f \subset [-N,N]\}$$

Each space

$$C^o_N \;=\; \{f \in C^o_c(\mathbb{R}) : \mathrm{spt} f \subset [-N,N]\} \subset C^o[-N,N]\}$$

is strictly smaller than the space $C^o[-N,N]$ of *all* continuous functions on the interval $[-N,N]$, since functions in C^o_N must vanish at the endpoints. Still, C^o_N is a *closed* subspace of the Banach space $C^o[-N,N]$ (with sup norm), since a sup-norm limit of functions vanishing at $\pm N$ must also vanish there. Thus, each individual C^o_N *is* a *Banach* space.

For $0 < M < N$ the space C^o_M is a *closed* subspace of C^o_N (with sup norm), since the property of vanishing off $[-M,M]$ is preserved under sup-norm limits.

But for $0 < M < N$ the space C^o_M is *nowhere dense* in C^o_N, since an open ball of radius $\varepsilon > 0$ around any function in C^o_N contains many functions with non-zero values off $[-M,M]$.

Thus, the *set* $C^o_c(\mathbb{R})$ is an ascending union of a countable collection of subspaces, each closed in its successor, but nowhere-dense there.

Though the topology on $C^o_c(\mathbb{R})$ is not specified yet, *any* acceptable topology on $C^o_c(\mathbb{R})$ should give the subspace C^o_M its natural (Banach-space) topology. *Then* $C^o_c(\mathbb{R})$ *is a countable union of nowhere-dense* *subsets.* By the Baire category theorem the topology on $C_c^o(\mathbb{R})$ cannot be complete metric. In particular, it cannot be Fréchet.

Nevertheless, the space $C_c^p(\mathbb{R})$ and many similarly-constructed spaces *do* have a reasonable structure, being an ascending union of a countable collection of Fréchet spaces, each closed in the next. [1]

[1.0.1] Remark: The space of integrals against regular Borel measures on a σ-compact[2] topological space X can be either *defined* or *proven*[3] depending on taste to be all continuous linear maps $C_c^o(X) \to \mathbb{C}$. This motivates understanding the topology of $C_c^o(X)$, and, thus, to understand non-Fréchet spaces.

[1.0.2] Remark: A similar argument shows that the space $C_c^\infty(\mathbb{R}^n)$ of *test functions*, compactly-supported infinitely differentiable functions, on $\mathbb{R}^n$ cannot be Fréchet. These functions play a central role in the study of *distributions* or *generalized functions*, providing further motivation to accommodate non-Fréchet spaces.

---

2. *Topological vectorspaces*

For the moment, the *scalars* need not be real or complex, need not be locally compact, and need not be commutative. Let k be a division ring. Any k-module V is a *free* k-module. [4] *We will substitute k-vectorspace for k-module in what follows.*

Let the scalars k have a *norm* $|\,|$, a non-negative real-valued function on k such that

$$\begin{cases} |x| = 0 \implies x = 0 \\ |xy| \;=\; |x||y| \\ |x+y| \;\; \le |x| + |y| \end{cases} \qquad \text{(for all } x, y \in k)$$

Further, suppose that with regard to the metric

$$d(x,y) \;=\; |x-y|$$

the topological space k is *complete* and *non-discrete.* The non-discreteness is that, for every $\varepsilon > 0$ there is $x \in k$ such that

$$0 < |x| < \varepsilon$$

A *topological vector space* V (over k) is a k-vectorspace V with a topology on V in which *points are closed*, and so that *scalar multiplication*

$$x \times v \longrightarrow xv \qquad \text{(for } x \in k \text{ and } v \in V)$$

and *vector addition*

$$v \times w \longrightarrow v + w \qquad \text{(for } v, w \in V)$$

[1] A countable ascending union of Fréchet spaces, each closed in the next, suitably topologized, is an *LF-space.* This stands for *limit of Fréchet.* The topology on the union is a *colimit*, discussed a bit later.

[2] As usual, *σ-compact* means that the space is a countable union of compacts.

[3] This is the Riesz-Markov-Kakutani theorem.

[4] The proof of this free-ness is the same as the proof that a vector space over a (commutative) field is free, that is, has a basis. The argument is often called the *Lagrange replacement principle*, and succeeds for infinite-dimensional vector spaces, granting the Axiom of Choice.

are *continuous*. For subsets X, Y of V, let

$$X + Y \;=\; \{x + y : x \in X, y \in Y\}$$

and

$$-X \;=\; \{-x : x \in X\}$$

The following idea is elementary, but indispensable. Given an open neighborhood U of 0 in a topological vectorspace V, continuity of vector addition yields an open neighborhood U' of 0 such that

$$U' + U' \;\subset\; U$$

Since $0 \in U'$, necessarily $U' \subset U$. This can be repeated to give, for any positive integer n, an open neighborhood U_n of 0 such that

$$\underbrace{U_n + \ldots + U_n}_{n} \;\subset\; U$$

In a similar vein, for fixed $v \in V$ the map $V \to V$ by $x \to x + v$ is a *homeomorphism*, being invertible by the obvious $x \to x - v$. Thus, *the open neighborhoods of* v *are of the form* $v + U$ *for open neighborhoods* U *of* 0. In particular, *a local basis at* 0 *gives the topology on a topological vectorspace.*

[2.0.1] Lemma: Given a compact subset K of a topological vectorspace V and a closed subset C of V not meeting K, there is an open neighborhood U of 0 in V such that

$$\text{closure}(K + U) \;\cap\; (C + U) \quad = \quad \phi$$

Proof: Since C is closed, for $x \in K$ there is a neighborhood U_x of 0 such that the neighborhood $x + U_x$ of x does not meet C. By continuity of vector addition

$$V \times V \times V \to V \quad \text{by} \quad v_1 \times v_2 \times v_3 \to v_1 + v_2 + v_3$$

there is a smaller open neighborhood N_x of 0 so that

$$N_x + N_x + N_x \;\subset\; U_x$$

By replacing N_x by $N_x \cap -N_x$, which is still an open neighborhood of 0, suppose that N_x is *symmetric* in the sense that $N_x = -N_x$.

Using this symmetry,

$$(x + N_x + N_x) \cap (C + N_x) \;=\; \phi$$

Since K is compact, there are finitely-many $x_1, \ldots, x_n$ such that

$$K \;\subset\; (x_1 + N_{x_1}) \cup \ldots \cup (x_n + N_{x_n})$$

Let

$$U \;=\; \bigcap_i N_{x_i}$$

Since the intersection is finite, this is open. Then

$$K + U \;\subset\; \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + U) \;\subset\; \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + N_{x_i})$$

These sets do not meet $C + U$, by construction, since $U \subset N_{x_i}$ for all i.

Finally, since $C + U$ is a union of opens $y + U$ for $y \in C$, it is open, so even the *closure* of $K + U$ does not meet $C + U$. ///

[2.0.2] Corollary: A topological vectorspace is *Hausdorff*. (Take $K = \{x\}$ and $C = \{y\}$ in the lemma). ///

[2.0.3] Corollary: The topological closure $\bar{E}$ of a subset E of a topological vectorspace V is obtained as

$$\bar{E} \;=\; \bigcap_U \, E + U$$

where U ranges over a local basis at 0.

Proof: In the lemma, take $K = \{x\}$ and $C = \bar{E}$ for a point x of V not in C. Then we obtain an open neighborhood U of 0 so that $x + U$ does not meet $\bar{E} + U$. The latter contains $E + U$, so certainly $x \notin E + U$. That is, for x not in the closure, there is an open U containing 0 so that $x \notin E + U$. ///

[2.0.4] Remark: It is convenient that Hausdorff-ness of topological vectorspaces follows from the weaker assumption that points are closed.

---

3. *Quotients and linear maps*

We continue to suppose that the *scalars* k are a *non-discrete complete normed division ring*. It suffices to think of $\mathbb{R}$ or $\mathbb{C}$.

As usual, for two topological vectorspaces V, W over k, a function

$$f : V \longrightarrow W$$

is *(k-)linear* when

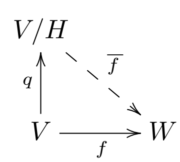
$$f(\alpha x + \beta y) \;=\; \alpha f(x) + \beta f(y)$$

for all $\alpha, \beta \in k$ and $x, y \in V$. Almost without exception we will be interested exclusively in *continuous* linear maps, meaning linear maps continuous for the topologies on V, W. The *kernel* $\ker f$ of a linear map is

$$\ker f \;=\; \{v \in V : f(v) = 0\}$$

Being the inverse image of a closed set by a continuous map, the kernel *closed*, and readily verified to be a k-subspace of V.

For a *closed* k-subspace H of a topological vectorspace V, the *quotient* V/H is a topological vectorspace with k-linear quotient map $q : V \to V/H$ through which any continuous $f : V \to W$ with $\ker f \supset H$ *factors*, in the sense that there is a unique continuous linear $\overline{f} : V/H \to W$ giving a commutative diagram



Uniqueness of the quotient $q : V \to V/H$, up to unique isomorphism, follows by the usual categorical arguments. The *existence* of the quotient is proven by the usual construction of V/H as the collection of cosets $v + H$, with q given as usual by

$$q : v \longrightarrow v + H$$

It is necessary to verify that this construction succeeds, as we do in the proposition below.

The *quotient topology* on V/H is necessarily the *finest* topology such that the quotient map $q : V \to V/H$ is continuous, namely, a subset E of V/H is open if and only if $q^{-1}(E)$ is open.

[3.0.1] Remark: For *non*-closed subspaces H, the quotient topology on the collection of cosets $\{v + H\}$ would *not* be Hausdorff. Thus, the proper categorical notion of topological vectorspace quotient, by non-closed subspace, would produce the collection of cosets $v + \overline{H}$ for the *closure* $\overline{H}$ of H. [5]

[3.0.2] Proposition: For a closed k-subspace W of a topological vectorspace V, the collection $Q = \{v+W : v \in V\}$ of cosets by W with map $q(v) = v + W$ is a topological vectorspace and q is a quotient map. In particular, points are *closed*.

Proof: The *algebraic* quotient $Q = V/W$ of cosets $v+W$ and $q(v) = v+W$ constructs a vectorspace quotient without any topological hypotheses on W. Since W is closed, and since vector addition is a homeomorphism, $v + W$ is closed as well. Thus, its complement $V - (v + W)$ is open, so $q(V - (v + W))$ is open, by definition of the quotient topology. Thus, the complement

$$q(v) \;=\; v + W \;=\; q(v + W) \;=\; V/W - q(V - (v + W))$$

of the open set $q(V - (v + W))$ is closed. ///

Unlike general topological quotient maps,

[3.0.3] Proposition: For a closed subspace H of a topological vector space V, the quotient map $q : V \to V/H$ is *open*, that is, carries open sets to open sets.

Proof: Let U be open in V. Then

$$q^{-1}(q(U)) \;=\; q^{-1}(U + H) \;=\; U + H \;=\; \bigcup_{h \in H} h + U$$

This is a union of opens, so is open. ///

[3.0.4] Corollary: Let $f : V \to X$ be a linear map with a closed subspace W of V contained in $\ker f$. Let $\bar{f}$ be the induced map $\bar{f} : V/W \to X$ defined by $\bar{f}(v + W) = f(v)$. Then f is continuous if and only if $\bar{f}$ is continuous.

Proof: Certainly if $\bar{f}$ is continuous then $f = \bar{f} \circ q$ is continuous. The converse follows from the fact that q is *open*. ///

This proves that the construction by cosets succeeds in producing a quotient: a continuous linear map $f : V \to X$ *factors through* any quotient V/W where W is a closed subspace contained in the kernel of f.

---

[5] That the coset-version of quotient V/H by a not-closed subspace H is not Hausdorff is easy to see, using the usual construticon-definition of the quotient topology, as follows. Let v be in the closure of H but not in H. Then every neighborhood U of v meets H. Every neighborhood of $v + H$ in the quotient is of the form $v + H + U$ for some neighborhood U of v in V, and includes 0. That is, even though the image of v in the quotient is not 0, every neighborhood of that image includes 0.

4. *More topological features*

The notions of *balanced subset, absorbing subset, directed set, Cauchy net*, and *completeness* are necessary. We continue to suppose that the *scalars* k are a *non-discrete complete normed division ring.*

A subset E of V is *balanced* if $xE \subset E$ for every $x \in k$ with $|x| \le 1$.

[4.0.1] Lemma: Every neighborhood u of 0 in a topological vectorspace V over k contains a *balanced* neighborhood N of 0.

Proof: By continuity of scalar multiplication, there is $\varepsilon > 0$ and a neighborhood U' of $0 \in V$ so that if $|x| < \varepsilon$ and $v \in U'$ then $xv \in U$. Since k is non-discrete, there is $x_o \in k$ with $0 < |x_o| < \varepsilon$. Since scalar multiplication by a non-zero element is a homeomorphism, $x_o U'$ is a neighborhood of 0 and $x_o U' \subset U$. Put

$$N = \bigcup_{|y|\le 1} y x_o U'$$

For $|x| \le 1$, $|xy| \le |y| \le 1$, so

$$xN = \bigcup_{|y|\le 1} x(yx_oU') \subset \bigcup_{|y|\le 1} yx_oU' = N$$

producing the desired N. ///

A subset E of vectorspace V over k is *absorbing* if for every $v \in V$ there is $t_o \in R$ so that $v \in \alpha E$ for every $\alpha \in k$ so that $|\alpha| \ge t_o$.

[4.0.2] Lemma: Every neighborhood U of 0 in a topological vectorspace is *absorbing.*

Proof: We may *shrink* U to assume U is *balanced.* By continuity of the map $k \to V$ given by $\alpha \to \alpha v$, there is $\varepsilon > 0$ so that $|\alpha| < \varepsilon$ implies $\alpha v \in U$. By the *non-discreteness* of k, there is non-zero $\alpha \in k$ satisfying any such inequality. Then $v \in \alpha^{-1} U$, as desired. ///

Let S be a *poset*, that is, a set with a partial ordering $\ge$. When for any two elements $s, t \in S$, there is $z \in S$ so that $z \ge s$ and $z \ge t$, S is a *directed set.*

A *net* in V is a subset $\{x_s : s \in S\}$ of V indexed by a directed set S. A net $\{x_s : s \in S\}$ in a topological vectorspace V is a *Cauchy net* if, for every neighborhood U of 0 in V, there is an index s_o so that for $s, t \ge s_o$ we have $x_s - x_t \in U$. A net $\{x_s : s \in S\}$ is *convergent* if there is $x \in V$ so that, for every neighborhood U of 0 in V there is an index s_o so that for $s \ge s_o$ we have $x - x_s \in U$. Since points are closed, there can be *at most* one point to which a net converges. Thus, *a convergent net is Cauchy.* A topological vectorspace is *complete* if (also) every Cauchy net is convergent.

[4.0.3] Lemma: Let Y be a vector subspace of a topological vector space X, *complete* when given the subspace topology from X. Then Y is a *closed* subset of X.

Proof: Let $x \in X$ be in the closure of Y. Let S be a local basis of opens at 0, where we take the partial ordering so that $U \ge U'$ if and only if $U \subset U'$. For each $U \in S$ choose

$$y_U \in (x + U) \cap Y$$

The net $\{y_U : U \in S\}$ converges to x, so is Cauchy. It must converge to a point in Y, so by uniqueness of limits of nets it must be that $x \in Y$. Thus, Y is closed. ///

[4.0.4] Remark: Unfortunately, *completeness* as above is too strong a condition for general topological vectorspaces, beyond Fréchet spaces. [6]

---

5. *Finite-dimensional spaces*

Finite-dimensional topological vectorspaces, and their interactions with other topological vectorspaces, are especially simple. [7] The central point is that *there is only one topology on a finite-dimensional space.*

[5.0.1] Proposition: For a one-dimensional topological vectorspace V, that is, a free module on one generator e, the map $k \to V$ by $x \to xe$ is a *homeomorphism.*

Proof: Since scalar multiplication is continuous, we need only show that the map is *open.* We need only do this at 0, since translation addresses other points. Given $\varepsilon > 0$, by the non-discreteness of k there is x_o in k so that $0 < |x_o| < \varepsilon$. Since V is Hausdorff, there is a neighborhood U of 0 so that $x_o e \notin U$. Shrink U so it is *balanced.* Take $x \in k$ so that $xe \in U$. For $|x| \geq |x_o|$, $|x_o x^{-1}| \leq 1$, so

$$x_o e \;=\; (x_o x^{-1})(xe) \in U$$

by the balanced-ness of U, contradiction. Thus,

$$xe \in U \implies |x| < |x_o| < \varepsilon$$

This proves the claim. ///

[5.0.2] Corollary: Fix $x_o \in k$. A not-identically-zero k-linear k-valued function f on V is *continuous* if and only if the *affine hyperplane*

$$H \;=\; \{v \in V : f(v) \;=\; x_o\}$$

is *closed* in V.

Proof: Certainly if f is continuous then H is closed. For the converse, we need only consider the case $x_o = 0$, since translations (i.e., vector additions) are homeomorphisms of V to itself.

For v_o with $f(v_o) \neq 0$ and for any other $v \in V$

$$f(v - f(v)f(v_o)^{-1}v_o) \;=\; f(v) - f(v)f(v_o)^{-1}f(v_o) \;=\; 0$$

Thus, V/H is one-dimensional. Let $\bar{f} : V/H \to k$ be the induced k-linear map on V/H so that $f = \bar{f} \circ q$:

$$\bar{f}(v + H) \;=\; f(v)$$

Then $\bar{f}$ is a homeomorphism to k, by the previous result, so f is continuous. ///

In the following theorem, the three assertions are proven together by induction on dimension.

[5.0.3] Theorem:
- A *finite-dimensional* k-vectorspace V has just one topological vectorspace topology.
- A finite-dimensional k-subspace V of a topological k-vectorspace W is necessarily a *closed* subspace of W.
- A k-linear map $\phi : X \to V$ to a finite-dimensional space V is continuous if and only if the kernel is closed.

---

[6] A slightly weaker version of completeness, *quasi-completeness* or *local* completeness, *does* hold for most important natural spaces, and will be discussed later.

[7] We still only need suppose that the scalar field k is a complete non-discrete normed division ring.

Proof: To prove the uniqueness of the topology, it suffices to prove that for any k-basis $e_1, \ldots, e_n$ for V, the map

$$k \times \ldots \times k \longrightarrow V$$

given by

$$(x_1, \ldots, x_n) \longrightarrow x_1 e_1 + \ldots + x_n e_n$$

is a homeomorphism. Prove this by induction on the dimension n, that is, on the number of generators for V as a free k-module.

The case $n = 1$ was treated already. Granting this, we need only further note that, since k is complete, the lemma above asserting the closed-ness of complete subspaces shows that any one-dimensional subspace is necessarily closed.

Take $n > 1$. Let

$$H = ke_1 + \ldots + ke_{n-1}$$

By induction, H is closed in V, so the quotient $q : V \to V/H$ can be constructed as expected, namely, as the set of cosets $v + H$. The space V/H is a one-dimensional topological vectorspace over k, with basis $q(e_n)$. By induction,

$$\phi : xq(e_n) = q(xe_n) \longrightarrow x$$

is a homeomorphism $V/H \to k$.

Likewise, ke_n is a closed subspace and we have the quotient map

$$q' : V \longrightarrow V/ke_n$$

The image has basis $q'(e_1), \ldots, q'(e_{n-1})$, and by induction

$$\phi' : x_1 q'(e_1) + \ldots + x_{n-1} q'(e_{n-1}) \to (x_1, \ldots, x_{n-1})$$

is a homeomorphism.

Invoking the induction hypothesis,

$$v \longrightarrow (\phi \circ q)(v) \times (\phi' \circ q')(v)$$

is continuous to

$$k^{n-1} \times k \approx k^n$$

On the other hand, by the continuity of scalar multiplication and vector addition,

$$k^n \longrightarrow V \quad \text{by} \quad x_1 \times \ldots \times x_n \longrightarrow x_1 e_1 + \ldots + x_n e_n$$

is continuous. These two maps are mutual inverses, proving that we have a homeomorphism.

Thus, a n-dimensional subspace is homeomorphic to k^n, so is complete, since a finite product of complete spaces is complete. By the closed-ness of complete subspaces, it is closed.

Continuity of a linear map $f : X \to k^n$ implies that the kernel $N = \ker f$ is closed. On the other hand, for N closed, the set of cosets $x + N$ constructs a quotient, and is a topological vectorspace of dimension at most n. Therefore, the induced map $\bar{f} : X/N \to V$ is unavoidably continuous. Then $f = \bar{f} \circ q$ is continuous, where q is the quotient map. This completes the induction step. ///