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Seminorms and locally convex spaces

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For all [1] our purposes, topological vector spaces are *locally convex*, in the sense of having a basis at 0 consisting of *convex* opens.

We prove below that a *separating family of seminorms* produces a locally convex topology.

Conversely, *every* locally convex topology is given by *separating families of semi-norms*: the seminorms are *Minkowski functionals* associated to a local basis at 0 of *balanced, convex* opens.

Giving the topology on a locally convex V by a family of seminorms exhibits V as a dense subspace of a projective limit of Banach spaces, with the subspace topology. For non-metrizable topologies, necessarily the indexing set for the limit has no countable cofinal subset.

There are natural topological vector spaces which are *not* Fréchet, but will be seen later to have reasonable *completeness* properties. One type consists of *ascending unions* of Fréchet spaces, each closed in the next, called *strict colimits of Fréchet spaces*, or *LF-spaces*. Examples: letting

$$\mathbb{C}^n \;=\; \{(z_1, z_2, \ldots, z_n, 0, 0, \ldots) : z_j \in \mathbb{C}\}$$

the ascending union $\mathbb{C}^\infty$ is a strict colimit of these Banach spaces $\mathbb{C}^n$:

$$\mathbb{C}^\infty \;=\; \bigcup_n \mathbb{C}^n \;=\; \mathrm{colim}_n\, \mathbb{C}^n$$

Similarly, and more obviously relevant to function theory, let

$$C^o_N(\mathbb{R}) \;=\; \{f \in C^o(\mathbb{R}) : \mathrm{spt} f \subset [-N, N]\}$$

Then the space of *compactly-supported* continuous functions $C^o_c(\mathbb{R})$ is a strict colimit of Banach spaces

$$C^o_c(\mathbb{R}) \;=\; \bigcup_N C^o_N(\mathbb{R}) \;=\; \mathrm{colim}_N\, C^o_N(\mathbb{R})$$

The space of L. Schwartz' *test functions* $\mathcal{D}(\mathbb{R}) = C^\infty_c(\mathbb{R})$ on $\mathbb{R}$ or $\mathbb{R}^n$ is a strict colimit of Fréchet spaces: with

$$C^\infty_N(\mathbb{R}) \;=\; \{f \in C^\infty(\mathbb{R}) : \mathrm{spt} f \subset [-N, N]\}$$

certainly

$$C^\infty_c(\mathbb{R}) \;=\; \bigcup_N C^\infty_N(\mathbb{R}) \;=\; \mathrm{colim}_N\, C^\infty_N(\mathbb{R})$$

Further, we will see natural ascending unions which are *not* strict in this sense, such as unions of negative-index Sobolev spaces. We will see later that these characterizations of topologies are *correct*, in the sense that the spaces are suitably complete, specifically, *quasi-complete*. Complications in the notion of *completeness* in trans-Fréchet spaces are also discussed later.

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[1] There is an exception: to illustrate the fact that *not* all topological vectorspaces are locally convex, the appendix briefly considers spaces ℓ^p with $0 < p < 1$, with topologies *not* locally convex. This is the only use of these examples.

1. *Topologies from seminorms*

Topologies given via *seminorms* on vectorspaces are described. These spaces are invariably *locally convex*, in the sense of having a local basis at 0 consisting of *convex* sets.

Let V be a complex vectorspace. A *seminorm* ν on V is a real-valued function on V so that

$$\begin{cases} \nu(x) \geq 0 & \text{for all } x \in V & \textit{(non-negativity)} \\ \nu(\alpha x) = |\alpha| \cdot \nu(x) & \text{for all } \alpha \in \mathbb{C},\ x \in V & \textit{(homogeneity)} \\ \nu(x+y) \leq \nu(x) + \nu(y) & \text{for all } x, y \in V & \textit{(triangle inequality)} \end{cases}$$

We allow the situation that $\nu(x) = 0$ yet $x \neq 0$. A *pseudo-metric* on a set X is a real-valued function d on $X \times X$ so that

$$\begin{cases} d(x,y) \geq 0 & \textit{(non-negativity)} \\ d(x,y) = d(y,x) & \textit{(symmetry)} \\ d(x,x) \leq d(x,y) + d(x,z) & \textit{(triangle inequality)} \end{cases}$$

We allow $d(x,y) = 0$ for $x \neq y$. The *associated pseudo-metric* attached to the seminorm ν is

$$d(x,y) \;=\; \nu(x-y) \;=\; \nu(y-x)$$

This pseudometric is a *metric* if and only if the seminorm is a *norm*.

Let $\{\nu_i : i \in I\}$ be a collection of semi-norms on a vectorspace V, with index set I. This family is a *separating family* of seminorms when for every $0 \neq x \in V$ there is ν_i so that $\nu_i(x) \neq 0$.

[1.0.1] Claim: The collection Φ of all *finite intersections* of sets

$$U_{i,\varepsilon} \;=\; \{x \in V : \nu_i(x) < \varepsilon\} \qquad (\text{for } \varepsilon > 0 \text{ and } i \in I)$$

is a *local basis* at 0 for a locally convex topology.

Proof: As expected, we intend to define a topological vector space topology on V by saying a set U is *open* if and only if for every $x \in U$ there is some $N \in \Phi$ so that

$$x + N \;\subset\; U$$

This would be the *induced topology* associated to the family of seminorms.

First, that we have a *topology* does not use the hypothesis that the family of seminorms is *separating*, although points will not be closed without the separating property. Arbitrary unions of sets containing 'neighborhoods' of the form $x + N$ around each point x have the same property. The empty set and the whole space V are visibly 'open'. The least trivial issue is to check that finite intersections of 'opens' are 'open'. Looking at each point x in a given finite intersection, this amounts to checking that finite intersections of sets in Φ are again in Φ. But Φ is *defined* to be the collection of all finite intersections of sets $U_{i,\varepsilon}$, so this works: we have closure under finite intersections, and we have a topology on V.

To verify that this topology makes V a topological vectorspace, we must verify the continuity of vector addition and continuity of scalar multiplication, and closed-ness of points. None of these verifications is difficult:

The *separating* property implies that the intersection of *all* the sets $x + N$ with $N \in \Phi$ is just x. Given a point $y \in V$, for each $x \neq y$ let U_x be an open set containing x but not y. Then

$$U \;=\; \bigcup_{x \neq y} U_x$$

is *open* and has complement $\{y\}$, so the singleton set $\{y\}$ is indeed *closed*.

To prove continuity of vector addition, it suffices to prove that, given $N \in \Phi$ and given $x, y \in V$ there are $U, U' \in \Phi$ so that

$$(x + U) + (y + U') \;\subset\; x + y + N$$

The triangle inequality for semi-norms implies that for a fixed index i and for $\varepsilon_1, \varepsilon_2 > 0$

$$U_{i,\varepsilon_1} + U_{i,\varepsilon_2} \;\subset\; U_{i,\varepsilon_1+\varepsilon_2}$$

Then

$$(x + U_{i,\varepsilon_1}) + (y + U_{i,\varepsilon_2}) \;\subset\; (x + y) + U_{i,\varepsilon_1+\varepsilon_2}$$

Thus, given

$$N \;=\; U_{i_1,\varepsilon_1} \cap \ldots \cap U_{i_n,\varepsilon_n}$$

take

$$U \;=\; U' \;=\; U_{i_1,\varepsilon_1/2} \cap \ldots \cap U_{i_n,\varepsilon_n/2}$$

proving continuity of vector addition.

For continuity of scalar multiplication, prove that for given $\alpha \in k$, $x \in V$, and $N \in \Phi$ there are $\delta > 0$ and $U \in \Phi$ so that

$$(\alpha + B_\delta) \cdot (x + U) \;\subset\; \alpha x + N \qquad\qquad (\text{with } B_\delta = \{\beta \in k : |\alpha - \beta| < \delta\})$$

Since N is an intersection of the special sub-basis sets $U_{i,\varepsilon}$, it suffices to consider the case that N is such a set. Given α and x, for $|\alpha' - \alpha| < \delta$ and for $x - x' \in U_{i,\delta}$,

$$\begin{aligned}\nu_i(\alpha x - \alpha' x') \;&=\; \nu_i((\alpha - \alpha')x + (\alpha'(x - x')) \;\leq\; \nu_i((\alpha - \alpha')x) + \nu_i(\alpha'(x - x')) \\ &=\; |\alpha - \alpha'| \cdot \nu_i(x) + |\alpha'| \cdot \nu_i(x - x') \;\leq\; |\alpha - \alpha'| \cdot \nu_i(x) + (|\alpha| + \delta) \cdot \nu_i(x - x') \\ &\leq\; \delta \cdot (\nu_i(x) + |\alpha| + \delta)\end{aligned}$$

Thus, to see the joint continuity, take $\delta > 0$ small enough so that

$$\delta \cdot (\delta + \nu_i(x) + |\alpha|) \;<\; \varepsilon$$

Taking finite intersections presents no further difficulty, taking the corresponding finite intersections of the sets B_δ and $U_{i,\delta}$, finishing the demonstration that separating families of seminorms give a structure of topological vectorspace.

Last, check that finite intersections of the sets $U_{i,\varepsilon}$ are convex. Since intersections of convex sets are convex, it suffices to check that the sets $U_{i,\varepsilon}$ themselves are convex, which follows from the homogeneity and the triangle inequality: with $0 \leq t \leq 1$ and $x, y \in U_{i,\varepsilon}$,

$$\nu_i(tx + (1-t)y) \;\leq\; \nu_i(tx) + \nu_i((1-t)y) \;=\; t\nu_i(x) + (1-t)\nu_i(y) \;\leq\; t\varepsilon + (1-t)\varepsilon \;=\; \varepsilon$$

Thus, the set $U_{i,\varepsilon}$ is convex. ///

2. *Seminorms from topologies: Minkowski functionals*

It takes a bit more work to go in the opposite direction, that is, to see that *every* locally convex topology is given by a family of seminorms.

Let U be a *convex* open set containing 0 in a topological vectorspace V. Every open neighborhood of 0 contains a *balanced* neighborhood of 0, so shrink U if necessary so it is balanced, that is, $\alpha v \in U$ for $v \in U$ and $|\alpha| \le 1$.

The *Minkowski functional* or *gauge* ν_U associated to U is

$$\nu_U(v) \;=\; \inf\{t \ge 0 : v \in tU\}$$

[2.0.1] Claim: The Minkowski functional ν_U associated to a balanced convex open neighborhood U of 0 in a topological vectorspace V is a *seminorm* on V, and is *continuous* in the topology on V.

Proof: The argument is as expected:

First, by continuity of scalar multiplication, *every* neighborhood U of 0 is *absorbing*, in the sense that every $v \in V$ lies inside tU for large enough $|t|$. Thus, the set over which we take the infimum to define the Minkowski functional is *non-empty*, so the infimum exists.

Let α be a scalar, and let $\alpha = s\mu$ with $s = |\alpha|$ and $|\mu| = 1$. The balanced-ness of U implies the balanced-ness of tU for any $t \ge 0$, so for $v \in tU$ also

$$\alpha v \;\in\; \alpha t U \;=\; s\mu t U \;=\; stU$$

From this,

$$\{t \ge 0 : \alpha v \in \alpha U\} \;=\; |\alpha| \cdot \{t \ge 0 : \alpha v \in tU\}$$

from which follows the *homogeneity* property required of a seminorm:

$$\nu_U(\alpha v) \;=\; |\alpha| \cdot \nu_U(v) \qquad\qquad \text{(for scalar } \alpha)$$

To prove the triangle inequality use the convexity. For $v, w \in V$ and $s, t > 0$ such that $v \in sU$ and $w \in tU$,

$$v + w \;\in\; sU + tU \;=\; \{su + tu' \;:\; u, u' \in U\}$$

By convexity,

$$su + tu' \;=\; (s+t) \cdot \Big(\frac{s}{s+t} \cdot u + \frac{t}{s+t} \cdot u'\Big) \;\in\; (s+t) \cdot U$$

Thus,

$$\nu_U(v+w) \;=\; \inf\{r \ge 0 : v + w \in rU\} \;\le\; \inf\{r \ge 0 : v \in rU\} + \inf\{r \ge 0 : w \in rU\} \;=\; \nu_U(v) + \nu_U(w)$$

Thus, the Minkowski functional ν_U attached to balanced, convex U is a continuous seminorm. ///

[2.0.2] Theorem: The topology of a *locally convex* topological vectorspace V is given by the collection of seminorms obtained as Minkowski functionals ν_U associated to a local basis at 0 consisting of convex, balanced opens.

Proof: The proof is straightforward. With or without local convexity, every neighborhood of 0 contains a *balanced* neighborhood of 0. Thus, a locally convex topological vectorspace has a local basis X at 0 of *balanced convex* open sets.

We claim that every open $U \in X$ can be recovered from the corresponding seminorm ν_U by

$$U \;=\; \{v \in V \;:\; \nu_U(v) < 1\}$$

Indeed, for $v \in U$, the continuity of scalar multiplication gives $\delta > 0$ and a neighborhood N of v such that $z \cdot v - 1 \cdot v \in U$ for $|1 - z| < \delta$. Thus, $v \in (1+\delta)^{-1} \cdot U$, so

$$\nu_U(v) \;=\; \inf\{t \geq 0 \;:\; v \in t \cdot U\} \;\leq\; \frac{1}{1+\delta} \;<\; 1$$

On the other hand, for $\nu_U(v) < 1$, there is $t < 1$ such that $v \in tU \subset U$, since U is convex and contains 0. Thus, the seminorm topology is at least as fine as the original.

Oppositely, the same argument shows that every seminorm local basis open

$$\{v \in V \;:\; \nu_U(v) < t\}$$

is simply tU. Thus, the original topology is at least as fine as the seminorm topology.

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[2.0.3] Remark: The above collection of seminorms is extravagantly large, since *all* convex balanced neighborhoods of 0 are used. Of course, there are relationships among these neighborhoods and the associated Minkowski functionals.

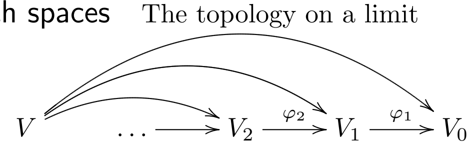
3. *Strong dual topologies and colimits*

The equality of the colimit topology on $H^{-\infty}(\mathbb{T}^n)$, with limitands $H^{-s}(T^n)$ with $-s \leq 0$ given their Hilbert space topologies, with the *strong dual topology* on $H^{-\infty}(\mathbb{T}^n)$ as dual to $H^{\infty}(\mathbb{T}^n)$, is inessential to proof of existence of tensor products and the Schwartz kernel theorem. Nevertheless, it is comforting to verify that this topology on $H^{-\infty}(\mathbb{T}^n)$ is the same as that described in another fashion, in terms of *seminorms*.

The instance of the *Schwartz Kernel Theorem* above refers to $H^{\infty}(\mathbb{T}^n)^* = H^{-\infty}(\mathbb{T}^n)$, the colimit/ascending union of $H^{-s}(\mathbb{T}^n) = H^s(\mathbb{T}^n)^*$ for $s \geq 0$. The strongest reasonable topology on each negative-index Levi-Sobolev space $H^{-s}(\mathbb{T}^n)$ is its Hilbert-space topology. As a vector space without topology, $H^{-\infty}(\mathbb{T}^n) = \bigcup_{s\geq 0} H^{-s}(\mathbb{T}^n)$. This ascending union is a *colimit*, which gives $H^{-\infty}(\mathbb{T}^n)$ a *topology*, naturally depending on the topologies of the limitands.

In fact, the argument below applies to limits of *Banach* spaces and colimits of their duals.

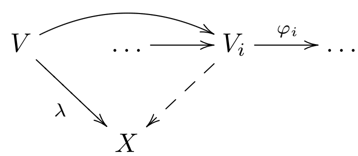
[3.1] Duals of limits of Banach spaces The topology on a limit



of Banach spaces V_i is given by the norms $|\cdot|_i$ on V_i, composed with the maps $\sigma_i : V \to V_i$, giving *seminorms* $p_i = |\cdot|_i \circ \sigma_i$. A collection of seminorms specifies a topology by giving a sub-basis for V at 0 consisting of opens of the form

$$U \;=\; \{v \in V \;:\; p_i(v) < \varepsilon\}$$

We recall the proof that linear maps $\lambda : V \to X$ from $V = \lim_i V_i$ of Banach spaces V_i to a normed space X necessarily factor through some limitand:



Proof: Without loss of generality, replace each V_i by the closure of the image of V_i in it. Continuity of λ is that, given $\varepsilon > 0$, there is an index i and a $\delta > 0$ such that

$$\lambda\Big(\{v \in V : p_i(v) < \delta\}\Big) \;\subset\; \{x \in X : |x|_X < \varepsilon\}$$

Then, for any $\varepsilon' > 0$,

$$\lambda\Big(\{v \in V : p_i(v) < \delta \cdot \frac{\varepsilon'}{\varepsilon}\}\Big) \;\subset\; \{x \in X : |x|_X < \varepsilon'\}$$

Thus, λ extends by continuity to the closure of $\sigma_i V$ in V_i, and gives a continuous map $V_i \to X$. ///

Thus, the *dual* of a limit of Banach spaces V_i is a colimit

$$V_0^* \xrightarrow{\varphi_1^*} V_1^* \xrightarrow{\varphi_1^*} V_2^* \longrightarrow \ldots \qquad \operatorname{colim} V_i^*$$

The duals V_i^* and the colimit are unambiguous as vector spaces. The topology on the colimit depends on the choice of topology on the duals V_i^*.

One reason to consider limits of Banach spaces V_i is the natural Banach-space structure on the dual. These are examples of *strong dual* topologies. In general, the *strong dual* topology on the dual V^* of a locally convex topological vector space V is given by seminorms [2]

$$p_E(\lambda) \;=\; \sup_{v \in E} |\lambda v| \qquad (E \text{ a } \textit{bounded}, \text{ convex, balanced neighborhood of } 0 \text{ in } V)$$

This gives a sub-basis at 0 for the topology on V^* consisting of sets

$$\{\lambda \in V^* \;:\; p_E(\lambda) < \varepsilon\} \qquad (\text{for } E \text{ bounded}, \varepsilon > 0)$$

where a *bounded* set E in a general topological vector space V is characterized by the property that, for every open neighborhood U of 0 in V, there is t_o such that $tU \supset E$ for all $t \geq t_o$.

Let $V = \lim V_i$ be a countable limit of Banach spaces, where all transition maps $V_i \to V_{i-1}$ are *injections*. We claim that the (locally convex) colimit $\operatorname{colim}_i\big(V_i^*\big)$ of the strong duals V_i^* gives the *strong dual* topology on the dual V^* of the limit $V = \lim V_i$.

Proof: Since the transition maps $V_i \to V_{i-1}$ are injections, as a set the limit V is the nested intersection of the V_i, and we identify V_i as a subset of V_{i-1}. Further, the dual V^* is identifiable with the ascending union of the duals V_i^*, regardless of topology.

The first point is to show that every bounded subset of V is contained in a bounded subset E expressible as a nested intersection of bounded subsets E_i of V_i. To see this, first note that the topology on V is given by the collection of (semi-) norms $|\cdot|_i$ on the individual Banach spaces V_i. A set $E \subset$ is bounded if and only if, for every index i, there is a radius r_i such that E is inside the ball $B_i(r_i)$ of radius r_i in V_i. We may as well replace these balls by the intersection of all the lower-(or-equal-)index balls:

$$E_i \;=\; \bigcap_{j \geq i} B_j(r_j)$$

The set E_i is bounded in V_i, $E_i \subset E_{i-1}$, and E is their nested intersection.

Now consider the linear functionals. On one hand, a given $\lambda : V \to \mathbb{C}$ factors through some $\lambda_i \in V_i^*$, and λE being inside the ε-ball B_ε in $\mathbb{C}$ is implied by $\lambda_i E_i \subset B_\varepsilon$ for some i. On the other hand, for $\lambda E \subset B_\varepsilon$,

[2]

we claim $\lambda E_i \subset B_\varepsilon$ for large-enough i. Indeed, λE_i is a balanced, bounded, convex subset of $\mathbb{C}$, so is a disk (either open or closed) of radius r_i. Since the intersection of the λE_i is inside B_ε, necessarily $\lim r_i \le \varepsilon$, with strict inequality if the disks are closed. Thus, there is i_o such that $r_i \le \varepsilon$ for $i \ge i_o$, with $r_i < \varepsilon$ for close disks. Thus, there is i_o such that $\lambda E_i \subset B_\varepsilon$ for $i \ge i_o$.

That is, the strong dual topology on $V^* = \bigcup_i V_i^*$ is the colimit of the strong dual (Banach) topologies on the V_i^*. ///

[3.1.1] Remark: The locally convex colimit of the Hilbert spaces $H^{-s}(\mathbb{T}^n)$ is $H^{-\infty}(\mathbb{T}^n)$, especially after verifying that the colimit topology from the strong duals $H^{-s}(\mathbb{T}^n)$ is the strong dual topology on $H^{+\infty}(\mathbb{T}^n)^*$.

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4. *Appendix: Non-locally-convex spaces* ℓ^p *with* $0 < p < 1$

With $0 < p < 1$, the topological vector space

$$\ell^p \;=\; \{\{x_i \in \mathbb{C}\} \;:\; \sum_i |x_i|^p \;<\; \infty\}$$

is not locally convex with the topology given by the metric $d(x,y) = |x-y|_p$ coming from

$$|x|_p \;=\; \sum_i |x_i|^p \qquad \text{(for } 0<p<1 \text{ no } p^{th} \text{ root!)}$$

It is *complete* with respect to this metric. Note that $|x|_p$ fails to be a *norm* by failing to be *homogeneous* of degree 1. The failure of local convexity is as follows.

Local convexity would require that the convex hull of the δ-ball at 0 be contained in some r-ball. That is, local convexity would require that, given δ, there is some r such that

$$\Big|\frac{1}{n}\cdot(\delta,0,\ldots) + \ldots + \frac{1}{n}\cdot(\underbrace{0,\ldots,0,\delta}_{n},0,\ldots)\Big|_p \;=\; \Big(\frac{\delta}{n}\Big)^p + \ldots + \Big(\frac{\delta}{n}\Big)^p \;<\; r \qquad \text{(for } n=1,2,3,\ldots)$$

That is, local convexity would require that, given δ, there is r such that

$$n^{1-p} \;<\; \frac{r}{\delta^p} \qquad \text{(for } n=1,2,3,\ldots)$$

This is impossible because $0<p<1$. ///

For contrast, to prove the triangle inequality for the alleged metric on ℓ^p with $0<p<1$, it suffices to prove that

$$(x+y)^p \;<\; x^p + y^p \qquad \text{(for } 0<p<1 \text{ and } x,y\ge 0)$$

To this end, take $x \ge y$. By the mean value theorem,

$$(x+y)^p \;\le\; x^p + p\xi^{p-1}y \qquad \text{(for some } x \le \xi \le x+y)$$

and

$$x^p + p\xi^{p-1}y \;\le\; x^p + px^{p-1}y \;\le\; x^p + py^{p-1}y \;=\; x^p + py^p$$
$$\le\; x^p + y^p \qquad \text{(since } p-1<0 \text{ and } \xi \ge x \ge y)$$

This proves the triangle inequality for $0<p<1$. ///
