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# *Hahn-Banach theorems*

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Convex sets can be separated by linear functionals. Second, continuous linear functionals on subspaces of a *locally convex*[1] topological vectorspace have continuous extensions to the whole space.

These assertions are proven first for *real* vectorspaces. The complex-linear versions are corollaries.

An important corollary is that in locally convex topological vectorspaces continuous linear functionals *separate points*, meaning that for $x \neq y$ there is a continuous linear functional $\lambda$ so that $\lambda(x) \neq \lambda(y)$.

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## 1. *Continuous Linear Functionals*

Let $k$ be either $\mathbf{R}$ or $\mathbf{C}$, and let $V$ be a $k$-vectorspace, without any assumptions about topologies for the moment. A $k$-linear $k$-valued function on $V$ is called a *linear functional*.

The space of all *continuous* $k$-linear $k$-valued functionals on $V$ is denoted by $V^*$, suppressing reference to the field $k$.

A linear functional $\lambda$ on $V$ is said to be *bounded* when there is a neighborhood $U$ of 0 in $V$ and constant $c$ so that $|\lambda x| \leq c$ for $x \in U$, where $|\ |$ is the usual absolute value on $k$. The following proposition is the general topological vectorspace analogue of the corresponding assertion for Banach spaces, in which boundedness has a different sense.

**[1.0.1] Proposition:** The following three conditions on a linear functional $\lambda$ on a topological vectorspace $V$ over $k$ are equivalent:
• $\lambda$ is continuous.
• $\lambda$ is continuous at 0
• $\lambda$ is bounded

*Proof:* The first assertion certainly implies the second. Assume the second. Then, given $\varepsilon > 0$, there is a neighborhood $U$ of 0 so that $|\lambda|$ is bounded by $\varepsilon$ on $U$. This proves boundedness. Finally, suppose that $|\lambda(x)| \leq c$ on a neighborhood $U$ of 0. Then given $x \in V$ and given $\varepsilon > 0$, we *claim* that for

$$y \in x + \frac{\varepsilon}{2c} U$$

we have

$$|\lambda(x) - \lambda(y)| < \varepsilon$$

Indeed, letting $x - y = \frac{\varepsilon}{2c} u$ with $u \in U$, we have

$$|\lambda(x) - \lambda(y)| = \frac{\varepsilon}{2c} |\lambda(u)| \leq \frac{\varepsilon}{2c} \cdot c = \frac{\varepsilon}{2} < \varepsilon$$

This proves the proposition. ///

[1] A topological vectorspace is *locally convex* when it has a local basis at 0 consisting of *convex* open sets. We are almost exclusively interested in locally convex spaces, and the hypothesis of local convexity is easily come-by.

# 2. *Dominated Extension Theorem*

In this section, all vectorspaces are *real*.

The result here involves only elementary algebra and inequalities (apart from transfinite induction) and is the heart of the matter. There is no direct discussion of topological vectorspaces here. The goal here is to *extend* a linear function while maintaining a comparison to another function (denoted $p$ below). Thus, for this section we need *not* suppose that the vectorspaces involved are *topological* vectorspaces.

Let $V$ be a *real* vectorspace, without any assumption about topologies. Let

$$p : V \;\to\; \mathbb{R}$$

be a *non-negative* real-valued function on $V$ so that

$$p(tv) \;=\; t \cdot p(v) \qquad (\text{for } t \geq 0) \qquad \textit{(positive-homogeneity)}$$

$$p(v + w) \leq p(v) + p(w) \qquad \textit{(triangle inequality)}$$

(Thus, $p$ is not quite a *semi-norm*, lacking any description of what happens to $p(tv)$ for $t < 0$).

**[2.0.1] Theorem:** Let $\lambda$ be a real-linear function on a real vector subspace $W$ of $V$, so that

$$\lambda(w) \;\leq\; p(w) \qquad\qquad (\text{for all } w \in W)$$

There is an extension of $\lambda$ to a real-linear function $\Lambda$ on all of $V$, so that

$$-p(-v) \leq \Lambda(v) \leq p(v)$$

for all $v \in V$.

*Proof:* The key is to extend the functional *one step*. That is, for $v_o \in V$, attempt to extend $\lambda'$ of $\lambda$ to $W + \mathbb{R}v_o$ by

$$\lambda'(w + tv_o) \;=\; \lambda(w) + ct$$

and examine the resulting conditions on $c$.

For all $w, w' \in W$

$$\lambda(w) - p(w - v_o) \;=\; \lambda(w + w') - \lambda(w') - p(w - v_o)$$
$$\leq\; p(w + w') - \lambda(w') - p(w - v_o) \;=\; p(w - v_o + w' + v_o) - \lambda(w') - p(w - v_o)$$
$$\leq\; p(w - v_o) + p(w' + v_o) - \lambda(w') - p(w - v_o) \;=\; p(w' + v_o) - \lambda(w')$$

That is,

$$\lambda(w) - p(w - v_o) \;\leq\; p(w' + v_o) - \lambda(w') \qquad\qquad (\text{for all } w, w' \in W)$$

Let $\sigma$ be the sup of all the left-hand sides as $w$ ranges over $W$. Since the right-hand side is finite, this sup is finite. With $\mu$ the inf of the right-hand side as $w'$ ranges over $W$,

$$\lambda(w) - p(w - v_o) \;\leq\; \sigma \;\leq\; \mu \;\leq\; p(w' + v_o) - \lambda(w')$$

Choose $c$ to be any real number so that

$$\sigma \;\leq\; c \;\leq\; \mu$$

and define

$$\lambda'(w + tv_o) \;=\; \lambda(w) + tc$$

To compare with $p$ is easy: in the inequality

$$\lambda(w) - p(w - v_o) \;\le\; \sigma$$

replace $w$ by $w/t$ with $t > 0$, multiply by $t$ and invoke the positive-homogeneity to obtain

$$\lambda(w) - p(w - tv_o) \;\le\; t\sigma$$

from which

$$\lambda'(w - tv_o) \;=\; \lambda(w) - tc \;\le\; \lambda(w) - t\sigma \;\le\; p(w - tv_o)$$

Likewise, from

$$\mu \;\le\; p(w + v_o) - \lambda(w)$$

a similar trick produces

$$\lambda'(w + tv_o) \;=\; \lambda(w) + tc \;\le\; \lambda(w) + t\mu \;\le\; p(w + tv_o)$$

for $t > 0$, the other half of the desired inequality.

Thus, for all $v \in W + Rv_o$ we have

$$\lambda'(v) \;\le\; p(v)$$

Using the linearity of $\lambda'$,

$$\lambda'(v) \;=\; -\lambda'(-v) \;\ge\; -p(-v)$$

which gives the bottom half of the comparison of $\lambda'$ and $p$.

To extend to a functional on the *whole* space dominated by $p$ is an exercise in transfinite induction, executed as follows. Let $\mathcal{X}$ be the collection of all pairs $(X, \mu)$, where $X$ is a subspace of $V$ (containing $W$), and where $\mu$ is real-linear real-valued function on $X$ so that $\mu$ restricted to $W$ is $\lambda$, and so that

$$-p(-x) \;\le\; \mu(x) \;\le\; p(x) \qquad\qquad \text{(for all } x \in X)$$

*Order* these by writing $(X, \mu) \le (Y, \nu)$ when $X \subset Y$ and $\nu|_X = \mu$. By the Hausdorff Maximality Principle, there is a *maximal* totally ordered subset $\mathcal{Y}$ of $\mathcal{X}$. Let

$$V' \;=\; \bigcup_{(X,\mu)\in\mathcal{Y}} X$$

be the ascending union of all the subspaces in $\mathcal{Y}$. Define a linear functional $\lambda'$ on this union as follows: for $v \in V'$, take any $X$ so that $(X, \mu) \in \mathcal{Y}$ and $v \in X$ and define

$$\lambda'(v) \;=\; \mu(v)$$

The total ordering on $\mathcal{Y}$ makes the choice of $(X, \mu)$ not affect the definition of $\lambda'$.

It remains to check that $V'$ is the whole space $V$. If not, the first part of the proof would create an extension to a properly larger subspace, contradicting the maximality. ///

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# 3. *Separation Theorem*

All vectorspaces are *real*.

Suppose that $V$ is a *locally convex* topological vectorspace, meaning that there is a local basis at $0 \in V$ consisting of convex sets. Invoking the dominated extension result just above we see that *open* convex sets can be 'separated' from convex sets:

**[3.0.1] Theorem:** For a non-empty convex open subset $X$ of a locally convex topological vectorspace $V$, and a non-empty convex set $Y$ in $V$ with $X \cap Y = \phi$, there is a *continuous* real-linear real-valued functional $\lambda$ on $V$ and a constant $c$ so that

$$\lambda(x) \;<\; c \;\leq\; \lambda(y) \qquad\qquad (\text{for all } x \in X \text{ and } y \in Y)$$

*Proof:* Fix $x_o \in X$ and $y_o \in Y$. Since $X$ is open, $X - x_o$ is open, and thus

$$U \;=\; (X - x_o) - (Y - y_o) \;=\; \{(x - x_o) - (y - y_o) : x \in X, \;\; y \in Y\}$$

is open. Further, since $x_o \in X$ and $y_o \in Y$, $U$ contains 0. Since $X, Y$ are convex, $U$ is convex. Shrink $U$ if necessary to assume that $U$ is *balanced*.

Define the *Minkowski functional* $p = p_U$ attached to $U$ by

$$p(v) \;=\; \inf\{t > 0 : v \in tU\}$$

The convexity assures that this function $p$ has the *positive-homogeneity* and *triangle-inequality* properties of the auxiliary functional $p$ mentioned in the dominated extension theorem above.

Let $z_o = -x_o + y_o$. Since $X \cap Y = \phi$, $z_o \notin U$, so $p(z_o) \geq 1$. Define a linear functional $\lambda$ on $\mathbb{R}z_o$ by

$$\lambda(tz_o) \;=\; t$$

Check that $\lambda$ is dominated by $p$ in the sense of the previous section:

$$\lambda(tz_o) \;=\; t \leq t \cdot p(z_o) \;=\; p(tz_o) \qquad\qquad (\text{for } t \geq 0)$$

while

$$\lambda(tz_o) \;=\; t \;<\; 0 \;\leq\; p(tz_o) \qquad\qquad (\text{for } t < 0)$$

Thus,

$$\lambda(tz_o) \;\leq\; p(tz_o) \qquad\qquad (\text{for all real } t)$$

Thus, $\lambda$ extends to a real-linear real-valued functional $\Lambda$ on $V$, still so that

$$-p(-v) \;\leq\; \Lambda(v) \;\leq\; p(v) \qquad\qquad (\text{for all } v \in V)$$

From the definition of $p$, since $U$ is balanced, $|\Lambda| \leq 1$ on $U$. Thus, on $\frac{\varepsilon}{2}U$ we have $|\Lambda| < \varepsilon$. That is, the linear functional $\Lambda$ is *bounded*, so is *continuous* at 0, so is *continuous* on $V$.

For arbitrary $x \in X$ and $y \in Y$,

$$\Lambda x - \Lambda y + 1 \;=\; \Lambda(x - y + z_o) \;\leq\; p(x - y + z_o) < 1$$

since $x - y + z_o \in U$. Thus, for all such $x, y$, we have

$$\Lambda x - \Lambda y \;<\; 0$$

Therefore, $\Lambda(X)$ and $\Lambda(Y)$ are *disjoint* convex subsets of $\mathbb{R}$. Since $\Lambda$ is not the zero functional, it is *surjective* to $\mathbb{R}$, and so is an *open* map. Thus, $\Lambda(X)$ is open, and

$$\Lambda(X) \;<\; \sup \Lambda(X) \;\leq\; \Lambda(Y)$$

as desired. ///

# 4. *Complex scalars*

The two theorems above, for *real* vectorspaces, have analogues in the complex case, which are really just corollaries of the real cases.

Let $V$ be a complex vectorspace. Given a complex-linear complex-valued functional $\lambda$ on $V$, let its real part be

$$u(v) \;=\; \mathrm{Re}\lambda(v) \;=\; \frac{\lambda(v) + \overline{\lambda(v)}}{2}$$

where the overbar denotes complex conjugation. On the other hand, given a *real*-linear *real*-valued functional $u$ on $V$, its *complexification* $Cu$ is

$$Cu(x) \;=\; u(x) - iu(ix)$$

where $i = \sqrt{-1}$.

**[4.0.1] Proposition:** For a real-linear functional $u$ on the complex vectorspace $V$, the complexification $Cu$ is a complex-linear functional so that

$$\mathrm{Re}(Cu) \;=\; u$$

and for a complex-linear functional $\lambda$

$$C(\mathrm{Re}\lambda) \;=\; \lambda$$

*(The proof is straightforward computation).*

**[4.0.2] Theorem:** Let $p$ be a *seminorm* on the complex vectorspace $V$. Let $\lambda$ be a complex-linear function on a complex vector subspace $W$ of $V$, so that

$$|\lambda(w)| \;\le\; p(w) \qquad\qquad (\text{for all } w \in W)$$

Then there is an extension of $\lambda$ to a complex-linear function $\Lambda$ on all of $V$, so that

$$|\Lambda(v)| \;\le\; p(v) \qquad\qquad (\text{for all } v \in V)$$

*Proof:* Certainly if $|\lambda| \le p$ then $|\mathrm{Re}\lambda| \le p$. By the theorem for *real*-linear functionals, there is an extension $u$ of $\mathrm{Re}\lambda$ to a *real*-linear functional so that still $|u| \le p$. Let

$$\Lambda \;=\; Cu$$

In light of the proposition, it remains to show that $|\Lambda| \le p$.

To this end, given $v \in V$, let $\mu$ be a complex number of absolute value 1 so that

$$|\Lambda(v)| \;=\; \mu\Lambda(v)$$

Then

$$|\Lambda(v)| \;=\; \mu\Lambda(v) \;=\; \Lambda(\mu v) \;=\; \mathrm{Re}\Lambda(\mu v) \;\le\; p(\mu v) \;=\; p(v)$$

where we use the seminorm property of $p$. Thus, the complex-linear functional made by complexifying the *real*-linear extension of the real part of $\lambda$ satisfies the desired bound. ///

**[4.0.3] Theorem:** Let $X$ be a non-empty convex open subset of a locally convex topological vectorspace $V$, and let $Y$ be an arbitrary non-empty convex set in $V$ so that $X \cap Y = \phi$. Then there is a *continuous* complex-linear complex-valued functional $\lambda$ on $V$ and a constant $c$ so that

$$\mathrm{Re}\lambda(x) \;<\; c \;\le\; \mathrm{Re}\lambda(y) \qquad\qquad (\text{for all } x \in X \text{ and } y \in Y)$$

*Proof:* Invoke the real-linear version of the theorem to make a *real*-linear functional $u$ so that

$$u(x) \;<\; c \;\leq\; u(y)$$

for all $x \in X$ and $y \in Y$. By the proposition, $u$ is the real part of its own complexification. ///

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# 5. *Corollaries*

The corollaries make sense for both real or complex scalars.

**[5.0.1] Corollary:** Let $V$ be a locally convex topological vectorspace. Let $K$ and $C$ be *disjoint* sets, where $K$ is a *compact* convex non-empty subset of $V$, and $C$ is a *closed* convex subset of $V$. Then there is a continuous linear functional $\lambda$ on $V$ and there are real constants $c_1 < c_2$ so that

$$\mathrm{Re}\,\lambda(x) \;\leq\; c_1 \;<\; c_2 \;\leq\; \mathrm{Re}\,\lambda(y) \qquad\qquad (\text{for all } x \in K \text{ and } y \in C)$$

*Proof:* Take a small-enough convex neighborhood $U$ of 0 in $V$ so that $(K+U) \cap C = \phi$. Apply the separation theorem to $X = K + U$ and $Y = C$. The constant $c_2$ can be taken to be $c_2 = \sup \mathrm{Re}\,\lambda(K+U)$. Since $\mathrm{Re}\,\lambda(K)$ is a compact subset of $\mathrm{Re}\,\lambda(K+U)$, its sup $c_1$ is strictly less than $c_2$. ///

**[5.0.2] Corollary:** Let $V$ be a locally convex topological vectorspace, $W$ a subspace, and $v_o \in V$. Let $\overline{W}$ denote the topological closure of $W$. Then $v_o \not\in \overline{W}$ if and only if there is a *continuous* linear functional $\lambda$ on $V$ so that $\lambda(W) = 0$ while $\lambda(v) = 1$.

*Proof:* On one hand, if $v_o$ lies in the closure of $W$, then any continuous function which is 0 on $W$ must be 0 on $v_o$, as well.

On the other hand, suppose that $v_o$ does *not* lie in the closure of $W$. Then apply the previous corollary with $K = \{v_o\}$ and $C = \overline{W}$. We find that

$$\mathrm{Re}\,\lambda(\{v_o\}) \cap \mathrm{Re}\,\lambda(\overline{W}) \;=\; \phi$$

Since $\mathrm{Re}\,\lambda(\overline{W})$ is a vector subspace of the real line, and is not the whole real line, it is just $\{0\}$, and $\mathrm{Re}\,\lambda(v_o) \neq 0$. Divide $\lambda$ by the constant $\mathrm{Re}\,\lambda(v_o)$ to obtain a continuous linear functional zero on $W$ but 1 on $v_o$. ///

**[5.0.3] Corollary:** Let $V$ be a locally convex topological (real) vectorspace. Let $\lambda$ be a continuous linear functional on a subspace $W$ of $V$. Then there is a continuous linear functional $\Lambda$ on $V$ extending $\lambda$.

*Proof:* Without loss of generality, take $\lambda \neq 0$. Let $W_o$ be the kernel of $\lambda$ (on $W$), and pick $w_1 \in W$ so that $\lambda w_1 = 1$. Evidently $w_1$ is not in the closure of $W_o$, so there is $\Lambda$ on the whole space $V$ so that $\Lambda|_{W_o} = 0$ and $\Lambda w_1 = 1$. It is easy to check that this $\Lambda$ is an extension of $\lambda$. ///

**[5.0.4] Corollary:** Let $V$ be a locally convex topological vectorspace. Given two distinct vectors $x \neq y$ in $V$, there is a continuous linear functional $\lambda$ on $V$ so that $\lambda(x) \neq \lambda(y)$

*Proof:* The set $\{x\}$ is compact convex non-empty, and the set $\{y\}$ is closed convex non-empty, so we can apply a corollary just above. ///