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Completeness and quasi-completeness

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The *appropriate* general completeness notion for topological vector spaces is *quasi-completeness*. There *is* a stronger general notion of *completeness*, which proves to be too strong in general. For example, the appendix shows that weak-star duals of infinite-dimensional Hilbert spaces are *quasi-complete*, but never *complete* in the stronger sense. Quasi-completeness for Fréchet spaces is ordinary metric-space completeness.

Hilbert, Banach, Fréchet, and LF-spaces, as well as their weak-star duals, and other spaces of mappings such as the *strong* and *weak* operator topologies on mappings between Hilbert spaces, are quasi-complete.

One important application of quasi-completeness of a topological vector space V is existence of *Gelfand-Pettis integrals* of compactly-supported, continuous, V-valued functions, discussed later.

An LF-space is a countable ascending union of Fréchet spaces with each Fréchet subspace *closed* in the next. These are *strict inductive limits* or *strict colimits* of Fréchet spaces. This construction includes spaces $C_c^\infty(\mathbb{R}^n)$ of *test functions*. Weak-star duals of the LF spaces are also quasi-complete, legitimizing integration of *distribution-valued* functions, and application of Cauchy-Goursat complex function theory to holomorphic families of distributions, for example.

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1. *Products, limits, coproducts, colimits*

The intuitive idea of a topological vectorspace being an *ascending union* of subspaces should be made precise. Spaces of *test functions* on $\mathbb{R}^n$ are important prototypes, being ascending unions of Fréchet spaces without being Fréchet spaces themselves. [1]

[1.1] A categorical viewpoint *Nested intersections* are examples of *limits*, and limits are *closed subobjects* of the corresponding *products*, while *ascending unions* are examples of *colimits*, and colimits are *quotients* (by closed subobjects) of the corresponding *coproducts*. Thus, proof of existence of *products* gets us half-way to proof of existence of *limits*, and proof of existence of *coproducts* gets us half-way to existence of *colimits*.

In more detail: locally convex *products* of locally convex topological vector spaces are proved to *exist* by constructing them as products of topological spaces, with the product topology, with scalar multiplication and vector addition *induced*.

Given the existence of locally convex *products*, a *limit* of topological vector spaces X_α with compatibility maps $p^\alpha_\beta : X_\alpha \to X_\beta$ is proven to exist by constructing it as the (closed) subobject Y of the product X of

[1] A countable ascending union of complete metric topological vector spaces, each a proper closed subspace of the next, *cannot* be complete *metric*, because it is *presented* as a countable union of nowhere-dense closed subsets, contradicting the conclusion of the Baire Category Theorem. For example, the ascending union of the chain of natural inclusions $\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \ldots$ is of this type. Nevertheless, such spaces are *complete* in a useful sense, clarified in the sequel.

the X_α consisting of $x \in X$ meeting the compatibility conditions

$$p_\beta(x) \;=\; (p^\alpha_\beta \circ p_\alpha)(x) \qquad\qquad \text{(for all } \alpha < \beta\text{, where } p_\ell \text{ is the } \textit{projection } X \to X_\ell)$$

Since local convexity is preserved by these constructions, these constructions do take place inside the category of *locally convex* topological vector spaces.

[1.2] Coproducts and colimits

Locally convex coproducts X of topological vector spaces X_α are coproducts of the vector spaces X_α with the *diamond topology*, described as follows. [2] For a collection U_α of convex neighborhoods of 0 in the X_α, let

$$U \;=\; \text{convex hull in } X \text{ of the union of } j_\alpha(U_\alpha) \qquad\qquad (\text{with } j_\alpha : X_\alpha \to X \text{ the } \alpha^{th} \text{ canonical map})$$

The diamond topology has local basis at 0 consisting of such U. Thus, it is locally convex by construction. Closedness of points follows from the corresponding property of the X_α. Thus, *existence* of a *locallly convex* coproduct of locally convex spaces is assured by the *construction.* The *locally convex* colimit of the X_α with compatibility maps $j^\alpha_\beta : X_\alpha \to X_\beta$ is the *quotient* of the locally convex coproduct X of the X_α by the *closure* of the subspace Z spanned by vectors

$$j_\alpha(x_\alpha) - (j_\beta \circ j^\alpha_\beta)(x_\alpha) \qquad\qquad (\text{for all } \alpha < \beta \text{ and } x_\alpha \in X_\alpha)$$

Annihilation of these differences in the quotient forces the desired compatibility relations. Obviously, quotients of locally convex spaces are locally convex.

[1.3] Strict colimits

Let V be a locally convex topological vectorspace with a countable collection of subspaces

$$V_o \subset V_1 \subset V_2 \subset V_3 \subset \ldots \qquad\qquad (\text{proper containments, } V_i \textit{ closed} \text{ in } V_{i+1})$$

and

$$V \;=\; \bigcup_i V_i \;=\; \mathrm{colim}_i V_i$$

Such V is a *strict (locally convex) colimit* or *strict (locally convex) inductive limit* of the V_i, *strict* because V_i is closed in V_{i+1}. The fact that the collection of subspaces is *countable* and *well-ordered* is also a special feature. We saw above that locally convex colimits *exist.*

A strict colimit of *Fréchet* spaces is an *LF-space*, for *limit-of-Fréchet.*

[1.3.1] Example:

Identify $\mathbb{C}^n$ with a subspace of $\mathbb{C}^{n+1}$ by

$$(z_1, \ldots, z_n) \;\longrightarrow\; (z_1, \ldots, z_n, 0)$$

The ascending union is denoted $\mathbb{C}^\infty$, and is a strict colimit of the $\mathbb{C}^n$. It consists of eventually-vanishing sequences of complex numbers. Its topology is strictly finer than the subspace topology it would inherit by mapping to ℓ^2. Because each limitand is Fréchet, even Banach or Hilbert, it is an LF-space.

[1.3.2] Example:

The space of compactly-supported smooth functions

$$C^o_c(\mathbb{R}^n) \;=\; \text{compactly-supported continuous functions on } \mathbb{R}^n$$

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[2] The *product* topology of locally convex topological vector spaces is unavoidably locally convex, whether the product is in the category of locally convex topological vector spaces or in the larger category of not-necessarily-locally-convex topological vector spaces. However, *coproducts* behave differently: the locally convex coproduct of *uncountably many* locally convex spaces is *not* a coproduct in the larger category of not-necessarily-locally-convex spaces. This already occurs with an uncountable coproduct of *lines.*

is a strict colimit of the subspaces

$$C^o_{B_\ell}(\mathbb{R}^n) \;=\; \{f \in C^o(\mathbb{R}^n) \;:\; \mathrm{spt} f \subset \text{closed ball } B_\ell \text{ of radius } \ell\}$$

Each space $C^o_{B_\ell}(\mathbb{R}^n)$ is a Banach space, being a closed subspace of the Banach space $C^o(B_\ell)$ by further requiring vanishing of the functions on the boundary of B_ℓ. Each limitand is Banach, hence Fréchet, so $C^o_c(\mathbb{R}^n)$ is an LF-space

[1.3.3] Example: The space of *test functions*

$$C^\infty_c(\mathbb{R}^n) \;=\; \text{compactly-supported smooth functions on } \mathbb{R}^n$$

is a strict colimit of the subspaces

$$C^\infty_{B_\ell}(\mathbb{R}^n) \;=\; \{f \in C^\infty_c(\mathbb{R}^n) \;:\; \mathrm{spt} f \subset \text{closed ball } B_\ell \text{ of radius } \ell\}$$

Each space $C^\infty_{B_\ell}(\mathbb{R}^n)$ is a Fréchet space, being a closed subspace of the Fréchet space

$$C^\infty(B_\ell) \;=\; \text{smooth functions on } B_\ell \;=\; \lim_k C^k(B_\ell) \;=\; \bigcap_k C^k(B_\ell)$$

by further requiring vanishing of the functions and all derivatives on the boundary of B_ℓ. The space of test functions is an LF-space.

[1.3.4] Example: Not all useful colimits are *strict*. The ascending union

$$H^{-\infty}(S^1) \;=\; \bigcup_s H^{-s}(S^1) \;=\; \mathrm{colim}_s H^{-s}(S^1)$$

of Levi-Sobolev spaces on the circle

$$H^{-s}(S^1) \;=\; \text{completion of } C^\infty(S^1) \text{ with respect to } |\cdot|_{-s}$$

with L^2 Levi-Sobolev norm

$$|\sum_n c_n\, e^{inx}|^2_{H^{-s}} \;=\; \sum_n |c_n|^2 \cdot (1+n^2)^{-s}$$

is *not* a strict colimit. While it does have a countable cofinal subfamily, by taking $s \in \mathbb{Z}$, the inclusions $H^{-\ell}(S^1) \to H^{-\ell-1}(S^1)$ have *dense, non-closed* images, and these inclusions are *not* homeomorphisms to their images.

[1.3.5] Proposition: In a colimit $V = \mathrm{colim} V_i$ indexed by positive integers, if every transition $V_i \to V_{i+1}$ is *injective*, then every limitand V_i *injects* to the colimit V.

Proof: Certainly each V_i injects to $W = \bigcup_n V_n$. We will give W a locally convex topology so that every inclusion $V_i \to W$ is continuous. The universal property of the colimit produces a map from the colimit to W, so every V_i must inject to the colimit itself.

Give W a local basis $\{U\}$ at 0, by taking arbitrary convex opens $U_i \subset V_i$ containing 0, and letting U be the convex hull of $\bigcup_n U_n$. The injection $V_i \to W$ is continuous, because the inverse image of such $U \cap V_i$ contains U_i, giving continuity at 0.

To be sure that *points are closed* in W, given $0 \neq x \in V_{i_o}$, we find a neighborhood of 0 in W not containing x. By Hahn-Banach, there is a continuous linear functional λ_{i_o} on V_{i_o} such that $\lambda_{i_o}(x) \neq 0$. Without loss of generality, $\lambda_{i_o}(x) = 1$ and $|\lambda_{i_o}| = 1$. Use Hahn-Banach to extend λ_{i_o} to a continuous linear functional λ_n on V_n for every n, with $|\lambda_n| \leq 1$. Then $U_n = \{y \in V_n : |\lambda_n(y)| < 1\}$ is open in V_n and does not contain x. The convex hull of $\bigcup_n U_n$ is just $\bigcup_n U_n$ itself, so does not contain x. ///

2. *Boundedness and equicontinuity in strict colimits*

[2.1] Bounded sets in strict colimits A subset B of a topological vector space V is *bounded* when, for every open neighborhood N of 0 there is $t_o > 0$ such that $B \subset tN$ for every $t \geq t_o$.

[2.1.1] Proposition: Assume that V is a *locally convex strict colimit* of a countable well-ordered collection of closed subspaces V_i. A subset B of V is *bounded* if and only it lies inside some subspace V_i and is a bounded subset of V_i.

Proof: Suppose B does *not* lie in any V_i. Then there is a sequence $i_1, i_2, \ldots$ of positive integers and x_{i_ℓ} in $V_{i_\ell} \cap B$ with x_{i_ℓ} *not* lying in $V_{i_\ell - 1}$. Using the simple nature of the indexing set and the simple interrelationships of the subspaces V_i,

$$V = \bigcup_j V_{i_\ell}$$

In particular, without loss of generality, we may suppose that $i_\ell = \ell$.

By the Hahn-Banach theorem and induction, using the closedness of V_{i-1} in V_i, there are continuous linear functionals λ_i on V_i's such that $\lambda_i(x_i) = i$ and the restriction of λ_i to V_{i-1} is λ_{i-1}, for example. Since V is the colimit of the V_i, this collection of functionals exactly describes a unique compatible continuous linear functional λ on V.

But $\lambda(B)$ is *bounded* since B is bounded and λ is continuous, precluding the possibility that λ takes on all positive integer values at the points x_i of B. Thus, it could *not* have been that B failed to lie inside some single V_i. The strictness of the colimit implies that B is bounded as a subset of V_i, proving one direction of the equivalence. The other direction of the equivalence is less interesting. ///

[2.2] Equicontinuity on strict colimits A set E of continuous linear maps from one topological vectorspace X to another topological vectorspace Y is *equicontinuous* when, for every neighborhood U of 0 in Y, there is a neighborhood N of 0 in X so that $T(N) \subset U$ for every $T \in E$.

[2.2.1] Proposition: Let V be a strict colimit of a well-ordered countable collection of locally convex closed subspaces V_i. Let Y be a locally convex topological vectorspace. Let E be a set of continuous linear maps from V to Y. Then E is *equicontinuous* if and only if for each index i the collection of continuous linear maps $\{T|_{V_i} : T \in E\}$ is equicontinuous.

Proof: Given a neighborhood U of 0 in Y, shrink U if necessary so that U is convex and balanced. For each index i, let N_i be a convex, balanced neighborhood of 0 in V_i so that $T(N_i) \subset U$ for all $T \in E$. Let N be the convex hull of the union of the N_i. By the convexity of N, still $T(N) \subset U$ for all $T \in E$. By the construction of the diamond topology, N is an open neighborhood of 0 in the coproduct, hence in the colimit, giving the equicontinuity of E. The other direction of the implication is easy. ///

3. *Banach-Steinhaus/uniform-boundedness*

It is easy to extend Banach-Steinhaus/uniform-boundedness to a class of situations including Fréchet spaces and LF-spaces. As a corollary, *joint* continuity of *separately* continuous bilinear maps on Fréchet spaces is obtained.

[3.0.1] Theorem: Let X be a Fréchet space or LF-space and Y a locally convex topological vector space. A set E of linear maps $X \to Y$, such that every set $Ex = \{Tx : T \in E\}$ is *bounded* in Y, is *equicontinuous*.

Proof: First consider X Fréchet. Given a neighborhood U of 0 in Y, let $A = \bigcap_{T\in E} T^{-1}\overline{U}$. By assumption, $\bigcup_n nA = X$. Since X is not a countable union of nowhere dense subsets, at least one of the closed sets nA is has non-empty interior. Since (non-zero) scalar multiplication is a homeomorphism, A itself has non-empty interior, containing some $x + N$ for a neighborhood N of 0 and $x \in A$. For every $T \in E$,

$$TN \;\subset\; T\{a - x : a \in A\} \;\subset\; \{u_1 - u_2 \;:\; u_1, u_2 \in \overline{U}\} \;=\; \overline{U} - \overline{U}$$

By continuity of addition and scalar multiplication in Y, given an open neighborhood U_o of 0, there is U such that $\overline{U} - \overline{U} \subset U_o$. Thus, $TN \subset U_o$ for every $T \in E$, and E is equicontinuous.

For $X = \bigcup_i X_i$ an LF-space, this argument already shows that E restricted to each X_i is equicontinuous. As observed in the previous section, this gives equicontinuity on the strict colimit. ///

[3.0.2] Remark: The idea of the proof of the theorem gives more than the statement of the theorem, although it is hard to make practical use of it. Namely, without assuming that X is Fréchet or LF, the argument shows that when

$$\{x \in X : Ex \text{ is bounded in } Y\}$$

is *not* a countable union of nowhere-dense subsets, then it is the whole X and E is equicontinuous.

[3.0.3] Corollary: Let $\beta : X \times Y \to Z$ be a bilinear map on Fréchet spaces X, Y, Z, continuous in each variable separately. Then β is *jointly* continuous.

Proof: Fix a neighborhood N of 0 in Z Let $x_n \to x_o$ in X and $y_n \to y_o$ in Y. For each $x \in X$, by continuity in Y, $\beta(x, y_n) \to \beta(x, y_o)$. Thus, for each $x \in X$, the set of values $\beta(x, y_n)$ is *bounded* in Z. The linear functionals $x \to \beta(x, y_n)$ are *equicontinuous*, by Banach-Steinhaus, so there is a neighborhood U of 0 in X so that $b_n(U) \subset N$ for all n. In the identity

$$\beta(x_n, y_n) - \beta(x_o, y_o) \;=\; \beta(x_n - x_o, y_n) + \beta(x_o, y_n - y_o)$$

we have $x_n - x_o \in U$ for large n, and $\beta(x_n - x_o, y_o) \in N$. Also, by continuity in Y, $\beta(x_o, y_n - y_o) \in N$ for large n. Thus, $\beta(x_n, y_n) - \beta(x_o, y_o) \in N + N$, proving *sequential* continuity. Since $X \times Y$ is metrizable, sequential continuity implies continuity. ///

4. *Ubiquity of quasi-completeness*

Since we will see that *quasi-completeness* is the appropriate general completeness notion, it is fortunate that most important topological vector spaces are quasi-complete. In particular, the later theorem on Gelfand-Pettis integrals will apply to all these.

Here we define *completeness* and *quasi-completeness* for general topological vector spaces, and show that quasi-completeness is preserved by important constructions on topological vector spaces. An appendix shows that the stronger *completeness* easily fails to hold, even for quasi-complete spaces.

Again, a subset B of a topological vector space V is *bounded* when, for every open neighborhood N of 0 there is $t_o > 0$ such that $B \subset tN$ for every $t \geq t_o$.

The notion of *directed set* generalizes the indexing set $\{1, 2, 3, \ldots\}$: a directed set I is a partially ordered set such that, given $i, j \in I$, there is $k \in I$ such that $k \geq i$ and $k \geq j$.

A *net* is a more general notion than *sequence*: while a *sequence* in a set S is essentially a map from $\{1, 2, 3, \ldots\}$ to S, a *net* in S is a map from a *directed set* to S.

A net $\{v_i : i \in I\}$ in a topological vector space V is a *converges* to $v_o \in V$ when, for every neighborhood N of 0 in V, there is $i_o \in I$ such that $v_i - v \in N$ for all $i \geq i_o$.

A net $\{v_i : i \in I\}$ in a topological vector space V is a *Cauchy net* when, for every neighborhood N of 0 in V, there is $i_o \in I$ such that $v_i - v_j \in N$ for all $i, j \geq i_o$.

A topological vector space is *complete* when every Cauchy *net* converges.

[4.0.1] Proposition: In metric spaces X, d sequential completeness implies completeness: Cauchy nets converge, and contain cofinal *sequences* converging to the same limit.

Proof: Indeed, let $\{s_i : i \in I\}$ be a Cauchy net in X. Given a natural number n, let $i_n \in I$ be an index such that $d(x_i, x_j) < \frac{1}{n}$ for $i, j \geq i_n$. Then $\{x_{i_n} : n = 1, 2, \ldots\}$ is a Cauchy sequence, with limit x. Given $\varepsilon > 0$, let $j \geq i_n$ be also large enough such that $d(x, x_j) < \varepsilon$. Then

$$d(x, x_{i_n}) \leq d(x, x_j) + d(x_j, x_{i_n}) < \varepsilon + \frac{1}{n} \qquad \text{(for every } \varepsilon > 0\text{)}$$

Thus, $d(x, x_{i_n}) \leq \frac{1}{n}$. We claim that the original Cauchy net also converges to x. Indeed, given $\varepsilon > 0$, for n large enough so that $\varepsilon > \frac{1}{n}$,

$$d(x_i, x) \leq d(x_i, x_{i_n}) + d(x_{i_n}, x) < \varepsilon + \varepsilon \qquad \text{(for } i \geq i_n\text{)}$$

with the strict inequality coming from $d(x_{i_n}, x) < \varepsilon$. ///

[4.0.2] Definition: A topological vectorspace is *quasi-complete* when every *bounded* Cauchy *net* is *convergent.*

Complete metric spaces are quasi-complete, so Hilbert, Banach, and Fréchet spaces are quasi-complete.

It is clear that *closed subspaces* of quasi-complete spaces are quasi-complete. Products and finite sums of quasi-complete spaces are quasi-complete.

[4.0.3] Proposition: A *strict colimit* of a *countable* collection of closed quasi-complete spaces is quasi-complete.

Proof: In the discussion of colimits we showed that bounded subsets of such colimits are exactly the bounded subsets of the limitands. Thus, bounded Cauchy nets in the colimit are bounded Cauchy nets in one of the closed subspaces. Each of these is assumed quasi-complete, so the colimit is quasi-complete. ///

Thus, spaces of test functions are quasi-complete, since they are such colimits of the Fréchet spaces of spaces of test functions with prescribed compact support.

[4.1] Quasi-completeness of spaces of linear maps

Let $\mathrm{Hom}^o(X, Y)$ be the space of continuous linear functions from a topological vectorspace X to another topological vectorspace Y. Give $\mathrm{Hom}^o(X, Y)$ the topology by seminorms $p_{x,U}$ where $x \in X$ and U is a convex, balanced neighborhood of 0 in Y, defined by [3]

$$p_{x,U}(T) \;=\; \inf\{t > 0 : Tx \in tU\} \qquad \text{(for } T \in \mathrm{Hom}^o(X, Y)\text{)}$$

[4.1.1] Theorem: For X a Fréchet space or LF-space, and Y quasi-complete, the space $\mathrm{Hom}^o(X, Y)$, with the topology induced by the seminorms $p_{x,U}$ where $x \in X$ and U is a convex, balanced neighborhood of 0 in Y, is *quasi-complete.*

[3] For X and Y Hilbert spaces, this construction of a topology on $\mathrm{Hom}^o(X, Y)$ gives the *strong operator topology* on $\mathrm{Hom}^o(X, Y)$. Replacing the topology on the Hilbert space Y by its *weak* topology, the construction gives $\mathrm{Hom}^o(X, Y)$ the *weak operator topology.* That the collection of continuous linear operators is the same in both cases is a consequence of the *Banach-Steinhaus theorem.*

[4.1.2] Remark: The starkest hypothesis on $\mathrm{Hom}^o(X,Y)$ is that it support the conclusion of the Banach-Steinhaus theorem. That is, a subset E of $\mathrm{Hom}^o(X,Y)$ so that the set of all images

$$Ex \;=\; \{Tx : T \in E\}$$

is *bounded* in Y for all $x \in X$ is *equicontinuous*. When X is Fréchet, this is true (by the usual Banach-Steinhaus theorem) for *any* Y. By the result above on bounded subsets of strict colimits, the same conclusion holds for X an LF-space.

Proof: Let $E = \{T_i : i \in I\}$ be a bounded Cauchy net in $\mathrm{Hom}^o(X,Y)$, where I is a directed set. Of course, attempt to define the limit of the net by

$$Tx \;=\; \lim_i T_i x$$

For $x \in X$ the evaluation map $S \to Sx$ from $\mathrm{Hom}^o(X,Y)$ to Y is continuous. In fact, the topology on $\mathrm{Hom}^o(X,Y)$ is the coarsest with this property. Therefore, by the quasi-completeness of Y, for each fixed $x \in X$ the net $T_i x$ in Y is bounded and Cauchy, so converges to an element of Y suggestively denoted Tx.

To prove *linearity* of T, fix x_1, x_2 in X, $a, b \in \mathbb{C}$ and fix a neighborhood U_o of 0 in Y. Since T is in the closure of E, for any open neighborhood N of 0 in $\mathrm{Hom}^o(X,Y)$, there exists

$$T_i \;\in\; E \cap (T + N)$$

In particular, for any neighborhood U of 0 in Y, take

$$N = \{S \in \mathrm{Hom}^o(X,Y) : S(ax_1 + bx_2) \in U,\; S(x_1) \in U,\; S(x_2) \in U\}$$

Then

$$\begin{aligned} &T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \\ &= \; (T(ax_1 + bx_2) - aT(x_1) - bT(x_2)) - (T_i(ax_1 + bx_2) - aT_i(x_1) - bT_i(x_2)) \end{aligned}$$

since T_i is linear. The latter expression is

$$\begin{aligned} &T(ax_1 + bx_2) - (ax_1 + bx_2) + a(T(x_1) - T_i(x_1) + b(T(x_2) - T_i(x_2) \\ &\in\; U + aU + bU \end{aligned}$$

By choosing U small enough so that

$$U + aU + bU \;\subset\; U_o$$

we find that

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \;\in\; U_o$$

Since this is true for every neighborhood U_o of 0 in Y,

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \;=\; 0$$

which proves linearity.

Continuity of the limit operator T exactly requires *equicontinuity* of $E = \{T_i x : i \in I\}$. Indeed, for each $x \in X$, $\{T_i x : i \in I\}$ is *bounded* in Y, so by Banach-Steinhaus $\{T_i : i \in I\}$ is equicontinuous.

Fix a neighborhood U of 0 in Y. Invoking the equicontinuity of E, let N be a small enough neighborhood of 0 in X so that $T(N) \subset U$ for all $T \in E$. Let $x \in N$. Choose an index i sufficiently large so that $Tx - T_i x \in U$, vis the definition of the topology on $\mathrm{Hom}^o(X,Y)$. Then

$$Tx \in U + T_i x \subset U + U$$

The usual rewriting, replacing U by U' such that $U' + U' \subset U$, shows that T is continuous. ///

The point here is to prove that the weak duals of reasonable topological vector spaces, such as infinite-dimensional Hilbert, Banach, or Fréchet spaces, are *not* complete. That is, in these weak duals there are Cauchy *nets* which *do not converge*.

The *incompleteness* (in the sense that not all Cauchy *nets* converge) of most weak duals has been known at least since [Grothendieck 1950], which gives a systematic analysis of completeness of various types of duals. This larger issue is systematically discussed in [Schaefer 1966/99], p. 147-8 and following.

5. *Appendix: Incompleteness of weak duals*

[5.0.1] Theorem: The weak dual of a locally-convex topological vector space V is *complete* if and only if every linear functional on V is *continuous*.

[5.0.2] Corollary: Weak duals of infinite-dimensional Hilbert, Banach, Fréchet, and LF-spaces are not *complete*. ///

[5.0.3] Remark: Nevertheless these duals are *quasi-complete*, which is sufficient for applications.

Proof: Overview of the proof: given a vector space V without a topology, we will topologize V with the finest-possible locally-convex topological vector space topology V_{init}. For this *finest* topology, all linear functionals *are* continuous, and the weak dual *is* complete in the strongest sense. Completeness of the weak dual of any coarser topology on V will be appraised by comparison to V_{init}.

Some set-up is necessary. Given a complex vector space V, let

$$V_{\text{init}} \;=\; \text{locally-convex colimit of finite-dimensional } X \subset V$$

with transition maps being inclusions.

[5.0.4] Claim: For a locally-convex topological vector space V the identity map $V_{\text{init}} \to V$ is continuous. That is, V_{init} is the finest locally convex topological vector space topology on V.

Proof: Finite-dimensional topological vector spaces have unique topologies. Thus, for any finite-dimensional vector subspace X of V the inclusion $X \to V$ is continuous with that unique topology on X. These inclusions form a compatible family of maps to V, so by the definition of colimit there is a *unique* continuous map $V_{\text{init}} \to V$. This map is the identity on every finite-dimensional subspace, so is the identity on the underlying set V. ///

[5.0.5] Claim: Every linear functional $\lambda : V_{\text{init}} \to \mathbb{C}$ is *continuous*.

Proof: The restrictions of a given linear function λ on V to finite-dimensional subspaces are compatible with the inclusions among finite-dimensional subspaces. Every linear functional on a finite-dimensional space is continuous, so the defining property of the colimit implies that λ is continuous on V_{init}. ///

[5.0.6] Claim: The weak dual V^* of a locally-convex topological vector space V injects continuously to the limit of the finite-dimensional Banach spaces

$$V^*_\Phi \;=\; \text{completion of } V^* \text{ under seminorm } p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad (\text{finite } \Phi \subset V)$$

and the weak dual topology is the subspace topology.

Proof: The weak dual topology on the continuous dual V^* of a topological vector space V is given by the seminorms

$$p_v(\lambda) \;=\; |\lambda(v)| \qquad (\text{for } \lambda \in V^* \text{ and } v \in V)$$

Specifically, a local sub-basis at 0 in V^* is given by sets

$$\{\lambda \in V^* \;:\; |\lambda(v)| < \varepsilon\}$$

The corresponding local basis is finite intersections

$$\{\lambda \in V^* \;:\; |\lambda(v)| < \varepsilon, \text{ for all } v \in \Phi\} \qquad (\text{for arbitrary finite sets } \Phi \subset V)$$

These sets contain, and are contained in, sets of the form

$$\{\lambda \in V^* \;:\; \sum_{v\in\Phi} |\lambda(v)| < \varepsilon\} \qquad \text{(for arbitrary finite sets } \Phi \subset V)$$

Therefore, the weak dual topology on V^* is also given by semi-norms

$$p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad (\text{finite } \Phi \subset V)$$

These have the convenient feature that they form a projective family, indexed by (reversed) inclusion. Let $V^*(\Phi)$ be V^* with the p_Φ-topology: this is not Hausdorff, so continuous linear maps $V^* \to V^*(\Phi)$ descend to maps $V^* \to V^*_\Phi$ to the *completion* V^*_Φ of V^* with respect to the pseudo-metric attached to p_Φ. The quotient map $V^*(\Phi) \to V^*_\Phi$ typically has a large kernel, since

$$\dim_{\mathbb{C}} V^*_\Phi \;=\; \mathrm{card}\,\Phi \qquad (\text{for finite } \Phi \subset V)$$

The maps $V^* \to V^*_\Phi$ are compatible with respect to (reverse) inclusion $\Phi \supset Y$, so V^* has a natural induced map to the $\lim_\Phi V^*_\Phi$. Since V separates points in V^*, V^* *injects* to the limit. The weak topology on V^* is exactly the subspace topology from that limit. ///

[5.0.7] Claim: The weak dual V^*_{init} of V_{init} is the limit of the finite-dimensional Banach spaces

$$V^*_\Phi \;=\; \text{completion of } V^*_{\mathrm{init}} \text{ under seminorm } \; p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad (\text{finite } \Phi \subset V)$$

Proof: The previous proposition shows that V^*_{init} *injects* to the limit, and that the subspace topology from the limit is the weak dual topology. On the other hand, the limit consists of linear functionals on V, without regard to topology or continuity. Since *all* linear functionals are continuous on V_{init}, the limit is naturally a subspace of V^*_{init}. ///

Returning to the proof of the theorem:

The limit $\lim_\Phi V^*_\Phi$ is a closed subspace of the corresponding *product*, so is *complete* in the strong sense. Any other locally convex topologization V_τ of V has weak dual $V^*_\tau \subset V^*_{\mathrm{init}}$ with the subspace topology, and is *dense* in V^*_{init}. Thus, unless $V^*_\tau = V^*_{\mathrm{init}}$, the weak dual V^*_τ *is not complete.* ///

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6. *Historical notes and references*

The fact that a bounded subset of a countable strict inductive limit of closed subspaces must actually be a bounded subset of one of the subspaces, easy to prove once conceived, is attributed to Dieudonne and Schwartz in [Horvath 1966]. See also [Bourbaki 1987], III.5 for this result. Pathological behavior of uncountable colimits was evidently first exposed in [Douady 1963].

Evidently *quotients* of quasi-complete spaces (by closed subspaces, of course) *may fail to be quasi-complete:* see [Bourbaki 1987], IV.63 exercise 10 for a construction.

The *incompleteness* of weak duals has been known at least since [Grothendieck 1950], which gives a systematic analysis of completeness of various types of duals. This larger issue is systematically discussed in [Schaefer 1966/99], p. 147-8 and following.

[Bourbaki 1987] N. Bourbaki, *Topological Vector Spaces, ch. 1-5*, Springer-Verlag, 1987.

[Douady 1963] A. Douady, *Parties compactes d'un espace de fonctions continues a support compact*, C. R. Acad. Sci. Paris 257, 1963, pp. 2788-2791.

[Grothendieck 1950] A. Grothendieck, *Sur la complétion du dual d'un espace vectoriel localement convexe*, C. R. Acad. Sci. Paris **230** (1950), 605-606.

[Horvath 1966] J. Horvath, *Topological Vector Spaces and Distributions*, Addison-Wesley, 1966.

[Schaefer 1966/99] H. Schaefer, *Topological vector spaces*, second edition with M.P. Wolff, Springer-Verlag, first edition 1966, second edition 1999.
