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Vector-valued integrals

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Quasi-complete, locally convex topological vector spaces V have the useful property that *continuous compactly-supported* V-valued functions have *integrals* with respect to finite Borel measures. Rather than *constructing* integrals as limits following [Bochner 1935], [Birkhoff 1935], *et alia*, we use the [Gelfand 1936]-[Pettis 1938] *characterization* of integrals, which has good functorial properties and gives a forceful reason for *uniqueness*. The issue is *existence*.

An immediate application is to decisive justification of differentiation with respect to a parameter inside an integral, under mild, easily understood hypotheses. This is a special case of a simple general assertion that Gelfand-Pettis integrals commute with continuous operators, described in the first section.

Another compelling application of this integration theory is to holomorphic vector-valued functions, with well-known application to the resolvents of operators on Hilbert and Banach spaces, as in [Dunford 1938] and [Taylor 1938]. In these sources Liouville's theorem on bounded entire $\mathbb{C}$-valued functions is invoked to prove that a bounded operator on a Hilbert or Banach spaces has non-empty spectrum.

Another of application of holomorphic and meromorphic vector-valued functions is to *generalized functions*, as in [Gelfand-Shilov 1964], studying *holomorphically parametrized families* of distributions. Many distributions which are not classical functions appear naturally as residues or analytic continuations of families of classical functions with a complex parameter.

A good theory of integration allows a natural treatment of convolutions of distributions with test functions, and related operations.

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1. *Gelfand-Pettis integrals and applications*

Let V be a complex topological vectorspace, f a measurable V-valued function on a measure space X. A *Gelfand-Pettis integral* of f is a vector $I_f \in V$ so that

$$\lambda(I_f) = \int_X \lambda \circ f \qquad \text{(for all } \lambda \in V^*)$$

If it exists and is unique, this vector I_f is denoted

$$I_f \;=\; \int_X f$$

In contrast to *construction* of integrals as limits of finite sums, this definition gives a *property* that no reasonable notion of integral would lack, without asking how the property comes to be. Since this property is an irreducible minimum, this characterization of integral is a *weak integral*.

Uniqueness of the integral is immediate when V^* *separates points* on V, as it does for *locally convex* V, by Hahn-Banach. Similarly, *linearity* of $f \to I_f$ follows when V^* separates points. Thus, the issue is *existence*.

[1]

The functions we integrate are relatively nice: compactly-supported and continuous, on measure spaces with *finite, positive, Borel* measures. In this situation, all the $\mathbb{C}$-valued integrals

$$\int_X \lambda \circ f$$

exist for elementary reasons, being integrals of compactly-supported $\mathbb{C}$-valued continuous functions on a compact set with respect to a finite Borel measure.

The technical requirement on the topological vectorspace V is that *the convex hull of a compact set has compact closure*. We show below that *quasi-completeness* and local convexity entail this property. Thus, for example, Hilbert, Banach, Fréchet, LF-spaces and their weak duals satisfy the hypothesis of the theorem.

[1.0.1] Theorem: Let X be a locally compact Hausdorff topological space with a *finite*, positive, Borel measure. Let V be a locally convex topological vectorspace in which the *closure of the convex hull of a compact set is compact*. Then *continuous compactly-supported* V-valued functions f on X have Gelfand-Pettis integrals. Further,

$$\int_X f \;\in\; \text{meas}\,(X)\cdot\Big(\text{closure of convex hull of } f(X)\Big)$$

[1.0.2] Remark: The conclusion that the integral of f lies in the closure of a convex hull, is a substitute for the estimate of a $\mathbb{C}$-valued integral by the integral of its absolute value.

[1.0.3] Corollary: Let $T: V \to W$ be a continuous linear map of locally convex topological vectorspaces, where convex hulls of compact sets in V have compact closures. Let f be a continuous, compactly-supported V-valued function on a finite regular measure space X. Then the W-valued function $T\circ f$ has a Gelfand-Pettis integral, and

$$T\Big(\int_X f\Big) \;=\; \int_X T\circ f$$

Proof: To verify that the left-hand side of the asserted equality fulfills the requirements of a Gelfand-Pettis integral of $T\circ f$, we must show that

$$\mu\Big(\text{left-hand side}\Big) \;=\; \int_X \mu\circ(T\circ f) \qquad\qquad (\text{for all } \mu \in W^*)$$

Starting with the left-hand side,

$$\begin{aligned}
\mu\Big(T\Big(\textstyle\int_X f\Big)\Big) &= (\mu\circ T)\Big(\textstyle\int_X f\Big) && (\text{associativity})\\
&= \textstyle\int_X (\mu\circ T)\circ f && (\mu\circ T \in V^* \text{ and } \textstyle\int_X f \text{ is a weak integral})\\
&= \textstyle\int_X \mu\circ(T\circ f) && (\text{associativity})
\end{aligned}$$

proving that $T\big(\int_X f\big)$ is a weak integral of $T\circ f$. ///

[1] We do require that the integral of a V-valued function be a vector in the space V itself, rather than in a larger space containing V, such as a double dual V^{**}, for example. Some alternative discussions of integration allow integrals to exist in larger spaces.

[1.0.4] Example: Differentiation under the integral can be justified coherently in terms of Gelfand-Pettis integrals, in many useful situations. For example, consider the Banach space $C^1\big([a,b]\times[c,d]\big)$ of real-differentiable functions of two real variables on $[a,b]\times[c,d]$. The norm is given by sup of values and sup of (norm of) derivative. The values of $f\in C^1\big([a,b]\times[c,d]\times\big)$ and (the norm) of its derivative are *uniformly* continuous, being continuous functions on a compact set. Thus, $t\to(x\to f(x,t))$ is a continuous $C^1[a,b]$-valued function on $[c,d]$.

By design, $T=\frac{\partial}{\partial t}$ is a continuous linear map from $V=C^1\big([a,b]\times[c,d]\big)$ to $W=C^o\big([a,b]\times[c,d]\big)$. By the corollary,

$$\frac{\partial}{\partial t}\int_a^b f(x,t)\,dx \;=\; \int_a^b \frac{\partial}{\partial t}f(x,t)\,dx$$

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2. *Proof of existence of Gelfand-Pettis integrals*

Again, *uniqueness* of Gelfand-Pettis integrals is clear, if they exist. Thus, the issue is proof of existence, by a construction.

Proof: To simplify, divide by a constant to make X have total measure 1. We may assume that X is *compact* since the support of f is compact. Let H be the closure of the convex hull of $f(X)$ in V, compact by hypothesis. We will show that there is an integral of f inside H.

For a *finite* subset L of V^*, let

$$V_L \;=\; \{v\in V: \lambda v=\int_X \lambda\circ f,\;\forall\lambda\in L\}$$

And let

$$I_L \;=\; H\cap V_L$$

Since H is compact and V_L is closed, I_L is *compact*. Certainly

$$I_L\cap I_{L'} \;=\; I_{L\cup L'}$$

for two finite subsets L,L' of V^*. Thus, if we prove that all the I_L are *non-empty*, then it will follow that the intersection of *all* these compact sets I_L is non-empty. (This is the *finite intersection property.*) That is, we will have *existence* of the integral.

To prove that each I_L is non-empty for *finite* subsets L of V^*, choose an ordering $\lambda_1,\ldots,\lambda_n$ of the elements of L. Make a continuous linear mapping $\Lambda=\Lambda_L$ from V to $\mathbb{R}^n$ by

$$\Lambda(v) \;=\; (\lambda_1 v,\ldots,\lambda_n v)$$

Since this map is continuous, the image $\Lambda(f(X))$ is compact in $\mathbb{R}^n$.

For a finite set L of functionals, the integral

$$y=y_L=\int_X \Lambda f(x)\,dx$$

is readily defined by component-wise integration. Suppose that this point y is in the convex hull of $\Lambda(f(X))$. Since Λ_L is linear, $y=\Lambda_L v$ for some v in the convex hull of $f(X)$. Then

$$\Lambda_L v \;=\; y \;=\; (\ldots,\int \lambda_i f(x)\,dx,\ldots)$$

Thus, the point v lies in I_L as desired. Granting that y lies in the convex hull of $\Lambda_L(f(x))$, we are done.

To prove that $y = y_L$ as above lies in the convex hull of $\Lambda_L(f(X))$, suppose *not.* From the lemma below, in a *finite-dimensional* space the convex hull of a compact set is still compact, *without* having to take closure. Thus, invoking also the finite-dimensional case of the Hahn-Banach theorem, there would be a linear functional η on $\mathbb{R}^n$ so that $\eta y > \eta z$ for all z in this convex hull. That is, letting $y = (y_1, \ldots, y_n)$, there would be real $c_1, \ldots, c_n$ so that for all $(z_1, \ldots, z_n)$ in the convex hull

$$\sum_i c_i z_i \;<\; \sum c_i y_i$$

In particular, for all $x \in X$

$$\sum_i c_i \lambda_i(f(x)) \;<\; \sum_i c_i y_i$$

Integration of both sides of this over X *preserves ordering*, giving the absurd

$$\sum_i c_i y_i < \sum_i c_i y_i$$

Thus, y *does* lie in this convex hull. ///

[2.0.1] Lemma: The convex hull of a compact set K in $\mathbb{R}^n$ is compact. In particular, we have compactness without taking closure.

Proof: We first claim that, for a set E in $\mathbb{R}^n$ and for any x a point in the convex hull of E, there are $n+1$ points $x_0, x_1, \ldots, x_n$ in E of which x is a convex combination.

By induction, to prove the claim it suffices to consider a convex combination $v = c_1 v_1 + \ldots + c_N v_N$ of vectors v_i with $N > n+1$ and show that v is actually a convex combination of $N-1$ of the v_i. Further, we can suppose without loss of generality that all the coefficients c_i are non-zero.

Define a linear map

$$L \;:\; \mathbb{R}^N \longrightarrow \mathbb{R}^n \times \mathbb{R} \qquad \text{by} \qquad L(x_1, \ldots, x_N) \longrightarrow (\sum_i x_i v_i,\; \sum_i x_i)$$

By dimension-counting, since $N > n+1$ the kernel of L must be non-trivial. Let $(x_1, \ldots, x_N)$ be a non-zero vector in the kernel.

Since $c_i > 0$ for every index, and since there are only finitely-many indices altogether, there is a constant c so that $|cx_i| \le c_i$ for every index i, and so that $cx_{i_o} = c_{i_o}$ for at least one index i_o. Then

$$v \;=\; v - 0 \;=\; \sum_i c_i v_i - c \cdot \sum x_i v_i \;=\; \sum_i (c_i - cx_i) v_i$$

Since $\sum_i x_i = 0$ this is still a convex combination, and since $cx_{i_o} = c_{i_o}$ at least one coefficient has become zero. This is the induction, which proves the claim.

Using the previous claim, a point v in the convex hull of K is actually a convex combination $c_o v_o + \ldots + c_n v_n$ of $n+1$ points $v_o, \ldots, v_n$ of K. Let σ be the compact set $(c_o, \ldots, c_n)$ with $0 \le c_i \le 1$ and $\sum_i c_i = 1$. The convex hull of K is the image of the compact set

$$\sigma \times K^{n+1}$$

under the continuous map

$$L \;:\; (c_o, \ldots, c_n) \times (v_o, v_1, \ldots, v_n) \longrightarrow \sum_i c_i v_i$$

so is compact. This proves the lemma, finishing the proof of the theorem. ///

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3. *Totally bounded sets in topological vectorspaces*

The point of this section is the last corollary, that *convex hulls of compact sets in Fréchet spaces have compact closures*. This is the key point for existence of Gelfand-Pettis integrals.

In preparation, we review the relatively elementary notion of *totally bounded subset* of a metric space, as well as the subtler notion of *totally bounded subset* of a topological vectorspace.

A subset E of a *complete metric space* X is *totally bounded* if, for every $\varepsilon > 0$ there is a covering of E by *finitely-many* open balls of radius ε. The property of *total boundedness* in a metric space is generally stronger than mere *boundedness*. It is immediate that any subset of a totally bounded set is totally bounded.

[3.0.1] Proposition: A subset of a complete metric space has compact closure if and only if it is *totally bounded*.

Proof: Certainly if a set has compact closure then it admits a finite covering by open balls of arbitrarily small (positive) radius.

On the other hand, suppose that a set E is totally bounded in a complete metric space X. To show that E has compact closure it suffices to show that any sequence $\{x_i\}$ in E has a Cauchy subsequence.

We choose such a subsequence as follows. Cover E by finitely-many open balls of radius 1. In at least one of these balls there are infinitely-many elements from the sequence. Pick such a ball B_1, and let i_1 be the smallest index so that x_{i_1} lies in this ball.

The set $E \cap B_1$ is still totally bounded (and contains infinitely-many elements from the sequence). Cover it by finitely-many open balls of radius $1/2$, and choose a ball B_2 with infinitely-many elements of the sequence lying in $E \cap B_1 \cap B_2$. Choose the index i_2 to be the smallest one so that both $i_2 > i_1$ and so that x_{i_2} lies inside $E \cap B_1 \cap B_2$.

Proceeding inductively, suppose that indices $i_1 < \ldots < i_n$ have been chosen, and balls B_i of radius $1/i$, so that

$$x_i \in E \cap B_1 \cap B_2 \cap \ldots \cap B_i$$

Then cover $E \cap B_1 \cap \ldots \cap B_n$ by finitely-many balls of radius $1/(n+1)$ and choose one, call it B_{n+1}, containing infinitely-many elements of the sequence. Let i_{n+1} be the first index so that $i_{n+1} > i_n$ and so that

$$x_{n+1} \in E \cap B_1 \cap \ldots \cap B_{n+1}$$

Then for $m < n$ we have

$$d(x_{i_m}, x_{i_n}) \le \frac{1}{m}$$

so this subsequence is Cauchy. ///

In a *topological vectorspace* V, a subset E is *totally bounded* if, for every neighborhood U of 0 there is a finite subset F of V so that

$$E \subset F + U$$

Here the notation $F + U$ means, as usual,

$$F + U = \bigcup_{v \in F} v + U = \{v + u : v \in F, u \in U\}$$

[3.0.2] Remark: In a topological vectorspace whose topology is given by a *translation-invariant* metric, a subset is *totally bounded* in this topological vectorspace sense if and only if it is totally bounded in the metric space sense, from the definitions.

[3.0.3] Lemma: In a topological vectorspace the convex hull of a *finite* set is *compact.*

Proof: Let the finite set be $F = \{x_1, \ldots, x_n\}$. Let σ be the compact set

$$\sigma \;=\; \{(c_1, \ldots, c_n) \in \mathbb{R}^n : \sum_i c_i = 1,\; 0 \le c_i \le 1, \;\text{ for all } i\} \;\subset\; \mathbb{R}^n$$

Then the convex hull of F is the continuous image of σ under the map

$$(c_1, \ldots, c_n) \to \sum_i c_i x_i$$

so is compact. ///

[3.0.4] Proposition: A totally bounded subset E of a *locally convex* topological vectorspace V has totally bounded *convex hull.*

Proof: Let U be a neighborhood of 0 in V. Let U_1 be a *convex* neighborhood of 0 so that $U_1 + U_1 \subset U$. Then for some finite subset F we have $E \subset F + U_1$, by the total boundedness. Let K be the convex hull of F, which by the previous result is *compact.* Then $E \subset K + U_1$, and the latter set is *convex*, as observed earlier. Therefore, the convex hull H of E lies inside $K + U_1$. Since K is compact, it is totally bounded, so can be covered by a finite union $\Phi + U_1$ of translates of U_1. Thus, since $U_1 + U_1 \subset U$,

$$H \subset (\Phi + U_1) + U_1 \subset \Phi + U$$

Thus, H lies inside this finite union of translates of U. This holds for any open U containing 0, so H is totally bounded. ///

[3.0.5] Corollary: In a Fréchet space, the closure of the convex hull of a compact set is compact.

Proof: A compact set in a Fréchet space (or in any complete metric space) is totally bounded, as recalled above. By the previous result, the convex hull of a totally bounded set in a Fréchet space (or in any locally convex space) is totally bounded. Thus, this convex hull has compact closure, since totally bounded sets in complete metric spaces have compact closure. ///

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4. *Quasi-completeness and convex hulls of compacts*

The following proof borrows an idea from the proof of the Banach-Alaoglu theorem. It reduces the general case to the case of Fréchet spaces, treated in the previous section.

[4.0.1] Proposition: In a quasi-complete locally convex topological vectorspace X, the closure C of the convex hull H of a compact set K is *compact.*

Proof: Since X is locally convex, by the Hahn-Banach theorem its topology is given by a collection of seminorms v. For each seminorm v, let X_v be the completion of the quotient

$$X/\{x \in X : v(x) = 0\}$$

with respect to the *metric* that v induces on the latter quotient. Thus, X_v is a Banach space. Consider

$$Z \;=\; \prod_v X_v \qquad\qquad \text{(with product topology)}$$

with the natural injection $j : X \to Z$, and with projection p_v to the v^{th} factor.

By construction, and by definition of the topology given by the seminorms, j is a (linear) homeomorphism to its image. That is, X is homeomorphic to the subset jX of Z, given the subspace topology.

The image $p_v jK$ is compact, being a continuous image of a compact subset of X. Since X_v is Fréchet, the convex hull H_v of $p_v jK$ has compact closure C_v. The convex hull jH of jK is contained in the product $\prod_v H_v$ of the convex hulls H_v of the projections $p_v jK$. By Tychonoff's theorem, the product $\prod_v C_v$ is *compact*.

Since jC is contained in the compact set $\prod_v C_v$, to prove that the closure jC of jH in jX is compact, it suffices to prove that jC is closed in Z. Since jC is a subset of the compact set $\prod_v C_v$, it is totally bounded and so is certainly bounded (in Z, hence in $X \approx jX$). By the quasi-completeness, any Cauchy net in jC converges to a point in jC. Since any point in the closure of jC in Z has a Cauchy net in jC converging to it, jC is closed in Z. This finishes the proof that quasi-completeness implies the compactness of closures of compact hulls of compacta. ///

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5. *Historical notes and references*

Most investigation and use of integration of vector-valued functions is in the context of *Banach-space*-valued functions. Nevertheless, the idea of [Gelfand 1936] extended and developed by [Pettis 1938] immediately suggests a viewpoint not confined to the Banach-space case. A hint appears in [Rudin 1991].

This is in contrast to many of the more detailed studies and comparisons of varying notions of integral specific to the Banach-space case, such as [Bochner 1935]. A variety of developmental episodes and results in the Banach-space-valued case is surveyed in [Hildebrandt 1953]. Proofs and application of many of these results are given in [Hille-Phillips 1957]. (The first edition, authored by Hille alone, is sparser in this regard.) See also [Brooks 1969] to understand the viewpoint of those times.

One of the few exceptions to the apparent limitation to the Banach-space case is [Phillips 1940]. However, it seems that in the United States after the Second World War consideration of anything fancier than Banach spaces was not popular.

The present pursuit of the issue of quasi-completeness (and compactness of the closure of the convex hull of a compact set) was motivated originally by the discussion in [Rudin 1991], although the latter does not make clear that this condition is fulfilled in more than Fréchet spaces, and does not mention quasi-completeness. Imagining that these ideas must be applicable to distributions, one might cast about for means to prove the compactness condition, eventually hitting upon the hypothesis of quasi-completeness in conjunction with ideas from the proof of the Banach-Alaoglu theorem. Indeed, in [Bourbaki 1987] it is shown (by apparently different methods) that quasi-completeness implies this compactness condition, although there the application to vector-valued integrals is not mentioned. [Schaeffer-Wolff 1999] is a very readable account of further important ideas in topological vector spaces.

The fact that a bounded subset of a countable strict inductive limit of closed subspaces must actually be a bounded subset of one of the subspaces, easy to prove once conceived, is attributed to Dieudonne and Schwartz in [Horvath 1966]. See also [Bourbaki 1987], III.5 for this result. Pathological behavior of uncountable colimits was evidently first exposed in [Douady 1963].

Evidently *quotients* of quasi-complete spaces (by closed subspaces, of course) *may fail to be quasi-complete:* see [Bourbaki 1987], IV.63 exercise 10 for a construction.

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