Holomorphic vector-valued functions

Paul Garrett  garrett@math.umn.edu  http://www.math.umn.edu/~garrett/

A first goal of classical complex function theory is Cauchy’s theorem, with Goursat’s refinement: complex differentiability implies the conclusion of Cauchy’s theorem, hence Cauchy’s integral formula, hence complex analyticity (expandability in power series). Thereafter, one can be casual about terminology, using complex-differentiable and analytic interchangeably. Use of the term holomorphic often signals completion of this basic Cauchy theory.

The same conclusions hold for vector-valued functions with values in a quasi-complete, locally convex topological vectorspace. This requires rethinking power series with coefficients in topological vector spaces, in addition to Gelfand-Pettis integrals and weak-to-strong principles.

- Fréchet spaces of holomorphic $\mathbb{C}$-valued functions
- Weak holomorphy implies strong holomorphy
- Appendix: Vector-valued power series, Abel’s theorem

1. Fréchet spaces of holomorphic $\mathbb{C}$-valued functions

[1.1] Topologies on holomorphic functions

The starting point of the Cauchy-Goursat ideas for scalar-valued holomorphic functions is Cauchy’s identity for complex-differentiable complex-valued $f$:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$$

where $\gamma$ is a simple closed curve circling $z$ once, counterclockwise, for example a circle with $z$ in its interior.

This suggests giving the space $\text{Hol}(\Omega)$ of $\mathbb{C}$-valued holomorphic functions on a non-empty open subset $\Omega \subset \mathbb{C}$ the locally convex topology by seminorms

$$\nu_K(f) = \sup_{z \in K} |f(z)| \quad (\text{for compact } K \subset \Omega)$$

This topology is also the natural topology on the space $C^0(\Omega)$ of continuous $\mathbb{C}$-valued functions on $\Omega$. Since $\Omega$ is a countable union of compacts, this topology is metrizable. We have seen that $C^0(\Omega)$ is complete in this topology, so is Fréchet. [1]

[1.1.1] Proposition: The space of holomorphic functions on $\Omega$ is complete with respect to the seminorms $\nu_K$, so is Fréchet.

Proof: Let $f_j$ be a sequence of holomorphic functions in $\Omega$, Cauchy in the $C^0(\Omega)$ topology. Thus, there is a uniform pointwise limit $f$ in $C^0(\Omega)$. From the Cauchy formula, for a simple closed curve $\gamma$ encircling $z$ just once,

$$f_j(z) = \frac{1}{2\pi i} \int_\gamma f_j(\zeta) \frac{d\zeta}{\zeta - z} \quad (\text{for } j = 1, 2, \ldots)$$

We claim that the same identity hold for the uniform-pointwise limit $f$. Using the Cauchy formula for $f_j$,
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\[ f(z) - \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta = f(z) + \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \left( f(z) - f_j(z) \right) - \frac{1}{2\pi i} \int_\gamma \frac{f_j(\zeta)}{\zeta - z} \]

\[ = f(z) - f_j(z) - \frac{1}{2\pi i} \int_\gamma \frac{f(z) - f_j(\zeta) d\zeta}{\zeta - z} \]

Since the image of \( \gamma \) is a continuous image of an interval \([a, b]\), it is compact, and the latter expression goes to 0 in \( j \).

Recall the straightforward argument for complex differentiability of functions given by such integrals: with small complex \( \varepsilon \neq 0 \),

\[ f(z + \varepsilon) - f(z) = \frac{1}{2\pi i} \int_\gamma \left( \frac{1}{\zeta - (z + \varepsilon)} - \frac{1}{\zeta - z} \right) f(\zeta) d\zeta \]

Of course,

\[ \frac{1}{\zeta - (z + \varepsilon)} - \frac{1}{\zeta - z} = \frac{\varepsilon}{(\zeta - (z + \varepsilon))(\zeta - z)} \]

so

\[ \frac{f(z + \varepsilon) - f(z)}{\varepsilon} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - (z + \varepsilon))(\zeta - z)} \]

The uniform continuity of \( \zeta \to 1/(\zeta - (z + \varepsilon)) \) gives

\[ \lim_{\varepsilon \to 0} \frac{f(z + \varepsilon) - f(z)}{\varepsilon} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) d\zeta}{(\zeta - z)^2} \]

In particular, the limit exists, so \( f \) is complex differentiable, and the space of holomorphic functions is complete. ///

Continuing the same argument gives

[1.1.2] Proposition: For any continuous function \( F \) defined on a curve \( \gamma \), the function

\[ f(z) = \frac{1}{2\pi i} \int_\gamma \frac{F(\zeta)}{\zeta - z} \]

is holomorphic, and

\[ f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{F(\zeta) d\zeta}{(\zeta - z)^{n+1}} \]

///

[1.1.3] Corollary: The Cauchy-Riemann differential operator \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \) is a continuous map \( \text{Hol}(\Omega) \to \text{Hol}(\Omega) \).

Proof: The Cauchy formula for the derivative

\[ f'(z) = \frac{\partial f}{\partial z} = \frac{1}{2\pi i} \int_\gamma \frac{F(\zeta) d\zeta}{(\zeta - z)^2} \]

with \( \gamma \) a circle of radius \( 2r \) inside \( \Omega \) centered at \( z_o \) bounds \( |f'(z)| \) on the disk \(|z - z_o| \leq r\) in terms of the sup of \(|f(\zeta)|\) on \( \gamma \):

\[ |f'(z)| \leq \frac{\text{length} \gamma}{2\pi} \cdot \frac{1}{(2r - r)^2} \cdot \sup_{|\zeta - z_o| = 2r} |f(\zeta)| \quad (\text{for } |z - z_o| \leq r) \]
Such a bound holds on some compact-closure neighborhood of every \( z \in \Omega \).

The continuity of \( \frac{\partial}{\partial z} \) on the locally convex, quasi-complete \( \text{Hol}(\Omega) \) allows invocation properties of Gelfand-Pettis/weak integrals. For example,

\[ 1.1.4 \text{ Corollary:} \] For a continuous, compactly-supported \( \text{Hol}(\Omega) \)-valued function \( w \to (z \to \Phi_w(z)) \) on a nice measure space \( W \),

\[ f(z) = \int_W \Phi_w(z) \, dw \]

is in \( \text{Hol}(\Omega) \), and

\[ \frac{\partial f}{\partial z} = \int_W \frac{\partial \Phi_w}{\partial z} \, dw \]

\textbf{Proof:} The integral of a continuous, compactly-supported, \( \text{Hol}(\Omega) \)-valued function \( w \to \Phi_w \) exists in \( \text{Hol}(\Omega) \), in the Gelfand-Pettis/weak sense. Thus, for any continuous linear map \( T \) to another locally convex topological vector space,

\[ T \left( \int_W \Phi_w \, dw \right) = \int_W T(\Phi_w) \, dw \]

In particular, this holds for \( T = \frac{\partial}{\partial z} : \text{Hol}(\Omega) \to \text{Hol}(\Omega) \).

A power series expansion near 0 is obtained from

\[ \frac{1}{\zeta - z} = \frac{1}{\zeta(1 - \frac{z}{\bar{\zeta}})} = \frac{1}{\bar{\zeta}} \left( 1 + \frac{z}{\bar{\zeta}} + \left( \frac{z}{\bar{\zeta}} \right)^2 + \ldots \right) \quad \text{(for } |z| < |\zeta| \text{)} \]

by estimating the tail of the infinite sum. For \(|\zeta| = 2r\) and \(|z| \leq r\),

\[ \frac{1}{1 - \frac{z}{\bar{\zeta}}} = \frac{1 - (\frac{z}{\bar{\zeta}})^{n+1}}{1 - \frac{z}{\bar{\zeta}}} + \left( \frac{z}{\bar{\zeta}} \right)^n + \left( \frac{z}{\bar{\zeta}} \right)^{n+1} \quad \text{(for } |z| < |\zeta| \text{)} \]

The remainder term is at most \( \frac{1}{2} \frac{n+1}{(2r - r)} \). Thus,

\[ f(z) = \sum_{k=0}^{n} z^k \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) \, d\zeta}{\zeta^{k+1}} + \frac{1}{2\pi i} \int_\gamma \left( \frac{z}{\bar{\zeta}} \right)^n \frac{f(\zeta) \, d\zeta}{1 - \frac{z}{\bar{\zeta}}} \]

The leftover term goes to 0 as \( n \to \infty \), and the coefficient of \( z^k \) is \( f^{(k)}(0)/k! \) from above.

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2. \textit{Weak holomorphy implies strong holomorphy}

Let \( V \) be a (almost certainly quasi-complete and locally convex) topological vector space and \( f \) a \( V \)-valued function on an open set \( \Omega \subset \mathbb{C} \).

A \( V \)-valued function \( f \) on an open subset \( \Omega \neq \phi \) of \( \mathbb{C} \) is (strongly) \textit{complex-differentiable} when

\[ \lim_{z \to z_0} \frac{1}{z - z_0} \cdot (f(z) - f(z_0)) \]

exists (in \( V \)) for all \( z_0 \in \Omega \), where \( z \to z_0 \) specifically means for \textit{complex} \( z \) approaching \( z_0 \).\footnote{The function \( f \) is (strongly) \textit{analytic} when it is locally expressible as a convergent power series with coefficients in \( V \).} The function \( f \) is \textit{weakly holomorphic} when the \( \mathbb{C} \)-valued functions \( \lambda \circ f \) are holomorphic for all \( \lambda \in V^* \).

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[2.0.1] **Theorem:** For $V$ a locally convex quasi-complete topological vector space, weakly holomorphic $V$-valued functions $f$ are strongly holomorphic. The usual Cauchy-theory integral formulas apply:

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

with $\gamma$ a closed path around $z$ having winding number $+1$. The function $f(z)$ is infinitely differentiable, in fact expressible as a convergent power series

$$f(z) = \sum_{n \geq 0} c_n (z - z_0)^n$$

with coefficients given by Cauchy’s formulas:

$$c_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

**Proof:** The first part of the proof, proving that weak holomorphy implies continuity, is parallel to the proof that weak $C^k[a,b]$ implies $C^{k-1}[a,b]$ for vector-valued functions.

To show that weak holomorphy of $f$ implies $f : D \to V$ is (strongly) continuous, without loss of generality prove continuity at $0$ and suppose $f(0) = 0 \in V$. Since $\lambda \circ f$ is holomorphic for each $\lambda \in V^*$ and vanishes at $0$, each function $(\lambda \circ f)(z)/z$ initially defined on a punctured disk at $0$ extends to a holomorphic function on a full disk at $0$. By Cauchy theory for the scalar-valued holomorphic function $z \to \frac{\lambda(\circ f)(\zeta)}{\zeta}$,

$$\frac{(\lambda \circ f)(z)}{z} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z - \zeta} \cdot \frac{(\lambda \circ f)(\zeta)}{\zeta} d\zeta$$

where $\gamma$ is a circle of radius $2r$ centered at $0$, and $|z| < r$. With $M_\lambda$ the sup of $|\lambda \circ f|$ on $\gamma$,

$$\left| \frac{(\lambda \circ f)(z)}{z} \right| \leq \frac{\text{length } \gamma}{2\pi} \cdot \frac{1}{2r} \cdot \frac{M_\lambda}{2r} = \frac{1}{2\pi} \cdot (2\pi \cdot 2r) \cdot \frac{1}{r} \cdot \frac{M_\lambda}{2r} = \frac{M_\lambda}{r}$$

Thus, the set of values

$$S = \left\{ \frac{f(z)}{z} : |z| \leq r \right\}$$

is weakly bounded. We have shown that weak boundedness implies (strong) boundedness, so $S$ is bounded. That is, given a balanced convex neighborhood $N$ of $0$ in $V$, there is $t_o > 0$ such that for complex $w$ with $|w| \geq t_o$, the set $S$ lies inside $wN$. Then

$$f(z) \in zwN$$

and $f(z) \in N$ for $|z| < |w|$. As $f(0) = 0$, we have proven that, given $N$, for $z$ sufficiently near $0$

$$f(z) - f(0) \in N$$

This is (strong) continuity.

Next, since $f(z)$ is (strongly) continuous, the integral

$$I(z) = \frac{1}{2\pi i} \int_{\gamma} f(\zeta) \frac{d\zeta}{\zeta - z}$$

is weakly bounded when it is bounded for the weak topology on $V$. That is, for each $\lambda \in V^*$, there is $t_\lambda > 0$ such that $|\lambda(v)| < t_\lambda^{-1}$ for every $v \in E$. 

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[3] Again, a set $E$ in $V$ is weakly bounded when it is bounded for the weak topology on $V$. That is, for each $\lambda \in V^*$, there is $t_\lambda > 0$ such that $|\lambda(v)| < t_\lambda^{-1}$ for every $v \in E$. 
exists as a Gelfand-Pettis integral. Thus, for any \( \lambda \in V^* \)

\[
\lambda(I(z)) = \frac{1}{2\pi i} \int_{\gamma} \frac{(\lambda \circ f)(\zeta)}{\zeta - z} \, d\zeta = (\lambda \circ f)(z)
\]

by the holomorphy of \( \lambda \circ f \). Since linear functionals separate points, \( I(z) = f(z) \), giving the Cauchy integral formula

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} \, d\zeta
\]

For the remainder of the discussion, there several possible ways to proceed: redo the Cauchy-Goursat for the vector-valued function possessing a Cauchy integral formula, or reduce vector-valued properties to scalar-valued. We use a combination.

To prove complex differentiability of \( f \) at \( z_0 \), take \( z_0 = 0 \) and use \( f(0) = 0 \), for convenience. There is a disk \( |z| < 3r \) such that for every \( \lambda \in V^* \)

\[
\lambda(F_{\lambda}(z)) = \frac{(\lambda \circ f)(z)}{z} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta} \frac{d\zeta}{\zeta - z}
\]

extends to a holomorphic function on \( |z| < r \). Continuity of \( f \) assures existence of

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta} \frac{d\zeta}{\zeta - z}
\]

By Cauchy theory for \( \mathbb{C} \)-valued functions, and Gelfand-Pettis,

\[
\lambda\left(\frac{f(z)}{z}\right) = F_{\lambda}(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{(\lambda \circ f)(\zeta)}{\zeta} \frac{d\zeta}{\zeta - z} = \lambda\left(\frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta} \frac{d\zeta}{\zeta - z}\right)
\]

Since functionals separate points,

\[
\frac{f(z)}{z} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta^2} \frac{d\zeta}{\zeta - z}
\]

From

\[
\frac{1}{\zeta(\zeta - z)} = \frac{1}{\zeta^2} + \frac{z}{\zeta^2(\zeta - z)}
\]

we have

\[
f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta^2} \, d\zeta + z \cdot \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta(\zeta - z)} \, d\zeta
\]

Using the continuity of \( f \), given a convex balanced neighborhood \( U \) of 0 in \( V \), the compact set

\[
K = \{ f(\zeta) : |\zeta| = 2r \}
\]

is contained in some multiple \( t_0 U \) of \( U \). Thus, for \( |z| < r \),

\[
\frac{f(z)}{z} - \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta^2} \, d\zeta \in |z| \cdot \frac{1}{(2r)^2} \cdot t_0 U
\]

and \( \lim_{z \to 0} f(z)/z \) exists. Since \( f(0) = 0 \), this proves the complex differentiability of \( f \):
3. Appendix: Vector-valued power series, Abel’s theorem

Here $V$ is a quasi-complete locally convex topological vector space.

[3.0.1] **Lemma:** Let $c_n$ be a **bounded** sequence of vectors in the locally convex quasi-complete topological vector space $V$. Let $z_n$ be a sequence of complex numbers, let $0 \leq r_n$ be real numbers such that $|z_n| \leq r_n$, and suppose that $\sum_n r_n < +\infty$. Then the series

$$\sum_n c_n z_n$$

*converges* in $V$. Further, given a convex balanced neighborhood $U$ of 0 in $V$ let $t$ be a positive real such that for all complex $\zeta$ with $|\zeta| \geq t$ we have $\{c_n\} \subset tU$. Then

$$\sum_n c_n z_n \in \left( \sum_n |z_n| \right) \cdot tU \subset \left( \sum_n r_n \right) \cdot tU$$

*Proof:* For convex balanced neighborhood $N$ of 0 in the topological vector space, with complex numbers $z$ and $w$ such that $|z| \leq |w|$, then $zN \subset wN$, since $|z/w| \leq 1$ implies $(z/w)N \subset N$, or $zN \subset wN$. Further, for an absolutely convergent series $\sum_n \alpha_n$ of complex numbers, for any $n_0$

$$\sum_{n \leq n_0} (\alpha_n \cdot V) = \sum_{n \leq n_0} (|\alpha_n| \cdot V) \subset \left( \sum_{n \leq n_0} |\alpha_n| \right) \cdot N \subset \left( \sum_{n < \infty} |\alpha_n| \right) \cdot N$$

For a balanced open $U$ containing 0, let $t$ be large enough such that for any complex $\zeta$ with $|\zeta| \geq t$ the sequence $c_n$ is contained in $\zeta U$. The previous discussion shows that

$$\sum_{m \leq t \leq n} c_\ell z_\ell \in (|z_m| + \ldots + |z_n|) \cdot tU$$

Given $\varepsilon > 0$, invoking absolute convergence, take $m$ sufficiently large such that for all $n \geq m$

$$|z_m| + \ldots + |z_n| < t \cdot \varepsilon$$

Then

$$\sum_{m \leq \ell \leq n} c_\ell z_\ell \in t \cdot (\varepsilon/t) \cdot U = U$$

Thus, the original series is convergent. Since $X$ is quasi-complete the limit exists in $V$. The last containment assertion follows from this discussion, as well. ///

[3.0.2] **Corollary:** Let $c_n$ be a bounded sequence of vectors in a locally convex quasi-complete topological vector space $V$. Then on $|z| < 1$ the series $f(z) = \sum_n c_n z^n$ converges and gives a **holomorphic** $V$-valued function. That is, the function is infinitely-many-times complex-differentiable.

*Proof:* The lemma shows that the series expressing $f(z)$ and its apparent $k^{th}$ derivative $\sum_n c_n \binom{n}{k} z^{n-k}$ all converge for $|z| < 1$. The usual direct proof of Abel’s theorem on the differentiability of (scalar-valued) power series can be adapted to prove the infinite differentiability of the $X$-valued function given by this power series, as follows. Let

$$g(z) = \sum_{n \geq 0} nc_n z^{n-1}$$
Then
\[
\frac{f(z) - f(w)}{z - w} - g(w) = \sum_{n \geq 1} c_n \left( \frac{z^n - w^n}{z - w} - nw^{n-1} \right)
\]

For \( n = 1 \), the expression in the parentheses is 1. For \( n > 1 \), it is
\[
(z^n - z^{n-1}w + \ldots + zw^{n-2} + w^{n-1}) - nw^{n-1}
\]
\[
= (z^{n-1} - w^{n-1}) + (z^{n-2}w - w^{n-1}) + \ldots + (zw^{n-3} - w^{n-1}) + (zw^{n-2} - w^{n-1}) + (w^{n-1} - w^{n-1})
\]
\[
= (z - w) \left[ (z^{n-2} + \ldots + w^{n-2}) + w(z^{n-3} + \ldots + w^{n-3}) + \ldots + w^{n-3}(z + w) + w^{n-2} + 0 \right]
\]
\[
= (z - w) \sum_{k=0}^{n-2} (k + 1) z^{n-2-k} w^k
\]

For \( |z| \leq r \) and \( |w| \leq r \) the latter expression is dominated by
\[
|z - w| \cdot r^{n-2} \frac{n(n - 1)}{2} < |z - w| \cdot n^2 r^{n-2}
\]

Let \( U \) be a balanced neighborhood of 0 in \( X \), and \( t \) a sufficiently large real number such that for all complex \( \zeta \) with \( |\zeta| \geq t \) all \( c_n \) lie in \( \zeta U \). For \( |z| \leq r < 1 \) and \( |w| \leq r < 1 \), by the lemma,
\[
\frac{f(z) - f(w)}{z - w} - g(w) = (z - w) \sum_{n \geq 2} c_n \cdot \left( \sum_{k=0}^{n-2} (k + 1) z^{n-2-k} w^k \right) \in (z - w) \cdot \left( \sum_n n^2 r^{n-2} \right) \cdot tU
\]

Thus, for any given convex balanced neighborhood \( U \) of 0 in \( X \), as \( z \to w \)
\[
\frac{f(z) - f(w)}{z - w} - g(w)
\]

eventually lies in \( U \).  

\[\text{[3.0.3] Corollary:}\] Let \( c_n \) be a sequence of vectors in a Banach space \( X \) such that for some \( r > 0 \) the series \( \sum |c_n| \cdot r^n \) converges in \( X \). Then for \( |z| < r \) the series \( f(z) = \sum c_n z^n \) converges and gives a holomorphic (infinitely-many times complex-differentiable) \( X \)-valued function.  