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Holomorphic vector-valued functions

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A first goal of classical complex function theory is Cauchy's theorem, with Goursat's refinement: complex differentiability implies the conclusion of Cauchy's theorem, hence Cauchy's integral formula, hence complex analyticity (expandability in power series). Thereafter, one can be casual about terminology, using *complex-differentiable* and *analytic* interchangeably. Use of the term *holomorphic* often signals completion of this basic Cauchy theory.

The same conclusions hold for vector-valued functions with values in a quasi-complete, locally convex topological vectorspace. This requires rethinking power series with coefficients in topological vector spaces, in addition to Gelfand-Pettis integrals and weak-to-strong principles.

- Fréchet spaces of holomorphic $\mathbb{C}$-valued functions
- Weak holomorphy implies strong holomorphy
- Appendix: Vector-valued power series, Abel's theorem

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1. *Fréchet spaces of holomorphic $\mathbb{C}$-valued functions*

[1.1] Topologies on holomorphic functions

The starting point of the Cauchy-Goursat ideas for scalar-valued holomorphic functions is Cauchy's identity for complex-differentiable complex-valued f:

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

where γ is a simple closed curve circling z once, counterclockwise, for example a circle with z in its interior.

This suggests giving the space $\mathrm{Hol}(\Omega)$ of $\mathbb{C}$-valued holomorphic functions on a non-empty open subset $\Omega \subset \mathbb{C}$ the locally convex topology by seminorms

$$\nu_K(f) \;=\; \sup_{z\in K} |f(z)| \qquad \text{(for compact } K \subset \Omega)$$

This topology is also the natural topology on the space $C^o(\Omega)$ of continuous $\mathbb{C}$-valued functions on Ω. Since Ω is a *countable* union of compacts, this topology is *metrizable*. We have seen that $C^o(\Omega)$ is *complete* in this topology, so is Fréchet. [1]

[1.1.1] Proposition: The space of holomorphic functions on Ω is *complete* with respect to the seminorms ν_K, so is Fréchet.

Proof: Let f_i be a sequence of holomorphic functions in Ω, Cauchy in the $C^o(\Omega)$ topology. Thus, there is a uniform pointwise limit f in $C^o(\Omega)$. From the Cauchy formula, for a simple closed curve γ encircling z just once,

$$f_j(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f_j(\zeta)\,d\zeta}{\zeta - z} \qquad \text{(for } j = 1, 2, \ldots)$$

We claim that the same identity hold for the uniform-pointwise limit f. Using the Cauchy formula for f_j,

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[1] In archaic terminology, a set $E \subset C^o(\Omega)$ is *normal* when it is pre-compact, that is, when its closure is compact.

$$f(z)-\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta-z} \;=\; f(z)+0-\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta-z} \;=\; \Big(f(z)-f_j(z)\Big)-\Big(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta-z}-\frac{1}{2\pi i}\int_\gamma \frac{f_j(\zeta)\,d\zeta}{\zeta-z}\Big)$$

$$=\; f(z)-f_j(z)-\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f_j(\zeta)\,d\zeta}{\zeta-z}$$

Since the image of γ is a continuous image of an interval $[a,b]$, it is *compact*, and the latter expression goes to 0 in j.

Recall the straightforward argument for complex differentiability of functions given by such integrals: with small complex $\varepsilon \neq 0$,

$$f(z+\varepsilon)-f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \Big(\frac{1}{\zeta-(z+\varepsilon)}-\frac{1}{\zeta-z}\Big)\,f(\zeta)\,d\zeta$$

Of course,

$$\frac{1}{\zeta-(z+\varepsilon)}-\frac{1}{\zeta-z} \;=\; \frac{\varepsilon}{(\zeta-(z+\varepsilon))\,(\zeta-z)}$$

so

$$\frac{f(z+\varepsilon)-f(z)}{\varepsilon} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{(\zeta-(z+\varepsilon))\,(\zeta-z)}$$

The uniform continuity of $\zeta \to 1/(\zeta-(z+\varepsilon))$ gives

$$\lim_{\varepsilon\to 0}\frac{f(z+\varepsilon)-f(z)}{\varepsilon} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{(\zeta-z)^2}$$

In particular, the limit exists, so f is complex differentiable, and the space of holomorphic functions is *complete*. ///

Continuing the same argument gives

[1.1.2] Proposition: For any continuous function F defined on a curve γ, the function

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{F(\zeta)\,d\zeta}{\zeta-z}$$

is holomorphic, and

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi i}\int_\gamma \frac{F(\zeta)\,d\zeta}{(\zeta-z)^{n+1}}$$

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[1.1.3] Corollary: The Cauchy-Riemann differential operator $\frac{\partial}{\partial z} = \frac{1}{2}\big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\big)$ is a continuous map $\mathrm{Hol}(\Omega) \to \mathrm{Hol}(\Omega)$.

Proof: The Cauchy formula for the derivative

$$f'(z) \;=\; \frac{\partial f}{\partial z} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{F(\zeta)\,d\zeta}{(\zeta-z)^2}$$

with γ a circle of radius $2r$ inside Ω centered at z_o bounds $|f'(z)|$ on the disk $|z-z_o| \le r$ in terms of the sup of $|f(\zeta)|$ on γ:

$$|f'(z)| \;\le\; \frac{\mathrm{length}\,\gamma}{2\pi}\cdot\frac{1}{(2r-r)^2}\cdot\sup_{|\zeta-z_o|=2r}|f(\zeta)| \qquad (\text{for } |z-z_o|\le r)$$

Such a bound holds on some compact-closure neighborhood of every $z \in \Omega$. ///

The continuity of $\frac{\partial}{\partial z}$ on the locally convex, quasi-complete $\mathrm{Hol}(\Omega)$ allows invocation properties of Gelfand-Pettis/weak integrals. For example,

[1.1.4] Corollary: For a continuous, compactly-supported $\mathrm{Hol}(\Omega)$-valued function $w \to (z \to \Phi_w(z))$ on a nice measure space W,

$$f(z) \;=\; \int_W \Phi_w(z)\, dw$$

is in $\mathrm{Hol}(\Omega)$, and

$$\frac{\partial f}{\partial z} \;=\; \int_W \frac{\partial \Phi_w}{\partial z}\, dw$$

Proof: The integral of a continuous, compactly-supported, $\mathrm{Hol}(\Omega)$-valued function $w \to \Phi_w$ exists in $\mathrm{Hol}(\Omega)$, in the Gelfand-Pettis/weak sense. Thus, for any continuous linear map T to another locally convex topological vector space,

$$T\Big(\int_W \Phi_w\, dw\Big) \;=\; \int_W T(\Phi_w)\, dw$$

In particular, this holds for $T = \frac{\partial}{\partial z} : \mathrm{Hol}(\Omega) \to \mathrm{Hol}(\Omega)$. ///

A power series expansion near 0 is obtained from

$$\frac{1}{\zeta - z} \;=\; \frac{1}{\zeta(1-\frac{z}{\zeta})} \;=\; \frac{1}{\zeta}\Big(1 + \frac{z}{\zeta} + \Big(\frac{z}{\zeta}\Big)^2 + \ldots\Big) \qquad (\text{for } |z| < |\zeta|)$$

by estimating the tail of the infinite sum. For $|\zeta| = 2r$ and $|z| \le r$,

$$\frac{1}{1-\frac{z}{\zeta}} \;=\; \frac{1-(\frac{z}{\zeta})^{n+1}}{1-\frac{z}{\zeta}} + \frac{(\frac{z}{\zeta})^{n+1}}{1-\frac{z}{\zeta}} \;=\; 1 + \frac{z}{\zeta} + \Big(\frac{z}{\zeta}\Big)^2 + \ldots + \Big(\frac{z}{\zeta}\Big)^n + \frac{(\frac{z}{\zeta})^{n+1}}{1-\frac{z}{\zeta}} \qquad (\text{for } |z| < |\zeta|)$$

The remainder term is at most $\frac{1}{2}^{n+1}/(2r-r)$. Thus,

$$f(z) \;=\; \sum_{k=0}^{n} z^k \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta^{k+1}} + \frac{1}{2\pi i}\int_\gamma \Big(\frac{z}{\zeta}\Big)^{n+1} \frac{f(\zeta)\, d\zeta}{1-\frac{z}{\zeta}}$$

The leftover term goes to 0 as $n \to \infty$, and the coefficient of z^k is $f^{(k)}(0)/k!$ from above. ///

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2. *Weak holomorphy implies strong holomorphy*

Let V be a (almost certainly quasi-complete and locally convex) topological vector space and f a V-valued function on an open set $\Omega \subset \mathbb{C}$.

A V-valued function f on an open subset $\Omega \neq \phi$ of $\mathbb{C}$ is (strongly) *complex-differentiable* when

$$\lim_{z \to z_o} \frac{1}{z - z_o} \cdot \big(f(z) - f(z_o)\big)$$

exists (in V) for all $z_o \in \Omega$, where $z \to z_o$ specificially means for *complex* z approaching z_o. [2] The function f is *weakly holomorphic* when the $\mathbb{C}$-valued functions $\lambda \circ f$ are holomorphic for all λ in V^*.

[2] The function f is (strongly) *analytic* when it is locally expressible as a convergent power series with coefficients in V.

[2.0.1] Theorem: For V a locally convex quasi-complete topological vector space, *weakly* holomorphic V-valued functions f are *strongly* holomorphic. The usual Cauchy-theory integral formulas apply:

$$f(z) \;=\; \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

with γ a closed path around z having winding number $+1$. The function $f(z)$ is infinitely differentiable, in fact expressible as a convergent power series

$$f(z) \;=\; \sum_{n\geq 0} c_n\,(z - z_o)^n$$

with coefficients given by Cauchy's formulas:

$$c_n \;=\; \frac{f^n(z_o)}{n!} \;=\; \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_o)^{n+1}}\, d\zeta$$

Proof: The first part of the proof, proving that weak holomorphy implies continuity, is parallel to the proof that weak $C^k[a,b]$ implies $C^{k-1}[a,b]$ for vector-valued functions.

To show that weak holomorphy of f implies $f : D \to V$ is (strongly) *continuous*, without loss of generality prove continuity at 0 and suppose $f(0) = 0 \in V$. Since $\lambda \circ f$ is holomorphic for each $\lambda \in V^*$ and vanishes at 0, each function $(\lambda \circ f)(z)/z$ initially defined on a punctured disk at 0 extends to a holomorphic function on a full disk at 0. By Cauchy theory for the scalar-valued holomorphic function $z \to \frac{\lambda(f(z))}{z}$,

$$\frac{(\lambda\circ f)(z)}{z} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\cdot\frac{(\lambda\circ f)(\zeta)}{\zeta}\, d\zeta$$

where γ is a circle of radius $2r$ centered at 0, and $|z| < r$. With M_λ the sup of $|\lambda\circ f|$ on γ,

$$\left|\frac{(\lambda\circ f)(z)}{z}\right| \;\leq\; \frac{\text{length}\,\gamma}{2\pi}\cdot\frac{1}{2r - r}\cdot\frac{M_\lambda}{2r} \;=\; \frac{1}{2\pi}\cdot(2\pi\cdot 2r)\cdot\frac{1}{r}\cdot\frac{M_\lambda}{2r} \;=\; \frac{M_\lambda}{r}$$

Thus, the set of values

$$S \;=\; \Big\{\frac{f(z)}{z} : |z| \leq r\Big\}$$

is *weakly* bounded. [3] We have shown that weak boundedness implies (strong) boundedness, so S is *bounded*. That is, given a balanced convex neighborhood N of 0 in V, there is $t_o > 0$ such that for complex w with $|w| \geq t_o$, the set S lies inside wN. Then

$$f(z) \;\in\; zwN$$

and $f(z) \in N$ for $|z| < |w|$. As $f(0) = 0$, we have proven that, given N, for z sufficiently near 0

$$f(z) - f(0) \;\in\; N$$

This is (strong) continuity.

Next, since $f(z)$ is (strongly) continuous, the integral

$$I(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

[3] Again, a set E in V is *weakly bounded* when it is *bounded* for the weak topology on V. That is, for each $\lambda \in V^*$, there is $t_\lambda > 0$ such that $|\lambda(v)| < t_\lambda^{-1}$ for every $v \in E$.

exists as a Gelfand-Pettis integral. Thus, for any $\lambda \in V^*$

$$\lambda\big(I(z)\big) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{(\lambda\circ f)(\zeta)}{\zeta - z}\,d\zeta \;=\; (\lambda\circ f)(z)$$

by the holomorphy of $\lambda\circ f$. Since linear functionals separate points, $I(z) = f(z)$, giving the Cauchy integral formula

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

For the remainder of the discussion, there several possible ways to proceed: redo the Cauchy-Goursat for the vector-valued function possessing a Cauchy integral formula, or reduce vector-valued properties to scalar-valued. We use a combination.

To prove complex differentiability of f at z_o, take $z_o = 0$ and use $f(0) = 0$, for convenience. There is a disk $|z| < 3r$ such that for every $\lambda \in V^*$

$$F_\lambda(z) \;=\; \frac{(\lambda\circ f)(z)}{z} \qquad\qquad (\text{on } 0 < |z| < r)$$

extends to a holomorphic function on $|z| < r$. Continuity of f assures existence of

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}\,\frac{d\zeta}{\zeta - z}$$

By Cauchy theory for $\mathbb{C}$-valued functions, and Gelfand-Pettis,

$$\lambda\Big(\frac{f(z)}{z}\Big) \;=\; F_\lambda(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{(\lambda\circ f)(\zeta)}{\zeta}\,\frac{d\zeta}{\zeta - z} \;=\; \lambda\Big(\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}\,\frac{d\zeta}{\zeta - z}\Big)$$

Since functionals separate points,

$$\frac{f(z)}{z} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}\,\frac{d\zeta}{\zeta - z}$$

From

$$\frac{1}{\zeta(\zeta - z)} \;=\; \frac{1}{\zeta^2} + \frac{z}{\zeta^2(\zeta - z)}$$

we have

$$\frac{f(z)}{z} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^2}\,d\zeta + z\cdot\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^2(\zeta - z)}\,d\zeta$$

Using the continuity of f, given a convex balanced neighborhood U of 0 in V, the compact set

$$K \;=\; \{f(\zeta) : |\zeta| = 2r\}$$

is contained in some multiple t_oU of U. Thus, for $|z| < r$,

$$\frac{f(z)}{z} - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^2}\,d\zeta \qquad\in\qquad |z|\cdot\frac{1}{(2r)^2\,r}\cdot t_oU$$

and $\lim_{z\to 0} f(z)/z$ exists. Since $f(0) = 0$, this proves the complex differentiability of f:

$$\lim_{z\to z_o} \frac{f(z) - f(z_o)}{z - z_o} \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_o)^2}\,d\zeta$$

The power series expansion is obtained as in the scalar-valued case. ///

3. *Appendix: Vector-valued power series, Abel's theorem*

Here V is a quasi-complete locally convex topological vector space.

[3.0.1] Lemma: Let c_n be a *bounded* sequence of vectors in the locally convex quasi-complete topological vector space V. Let z_n be a sequence of complex numbers, let $0 \le r_n$ be real numbers such that $|z_n| \le r_n$, and suppose that $\sum_n r_n < +\infty$. Then the series

$$\sum_n c_n\, z_n$$

converges in V. Further, given a convex balanced neighborhood U of 0 in V let t be a positive real such that for all complex ζ with $|\zeta| \ge t$ we have $\{c_n\} \subset tU$. Then

$$\sum_n c_n\, z_n \;\in\; \Big(\sum_n |z_n|\Big) \cdot tU \subset \Big(\sum_n r_n\Big) \cdot tU$$

Proof: For convex balanced neighborhood N of 0 in the topological vector space, with complex numbers z and w such that $|z| \le |w|$, then $zN \subset wN$, since $|z/w| \le 1$ implies $(z/w)N \subset N$, or $zN \subset wN$. Further, for an absolutely convergent series $\sum_n \alpha_n$ of complex numbers, for any n_o

$$\sum_{n \le n_o} (\alpha_n \cdot V) = \sum_{n \le n_o} (|\alpha_n| \cdot V) \subset \Big(\sum_{n \le n_o} |\alpha_n|\Big) \cdot N \subset \Big(\sum_{n < \infty} |\alpha_n|\Big) \cdot N$$

For a balanced open U containing 0, let t be large enough such that for any complex ζ with $|\zeta| \ge t$ the sequence c_n is contained in ζU. The previous discussion shows that

$$\sum_{m \le \ell \le n} c_\ell z_\ell \;\in\; (|z_m| + \ldots + |z_n|) \cdot tU$$

Given $\varepsilon > 0$, invoking absolute convergence, take m sufficiently large such that for all $n \ge m$

$$|z_m| + \ldots + |z_n| < t \cdot \varepsilon$$

Then

$$\sum_{m \le \ell \le n} c_\ell z_\ell \;\in\; t \cdot (\varepsilon/t) \cdot U = U$$

Thus, the original series is convergent. Since X is quasi-complete the limit exists in V. The last containment assertion follows from this discussion, as well. ///

[3.0.2] Corollary: Let c_n be a bounded sequence of vectors in a locally convex quasi-complete topological vector space V. Then on $|z| < 1$ the series $f(z) = \sum_n c_n z^n$ converges and gives a *holomorphic* V-valued function. That is, the function is infinitely-many-times complex-differentiable.

Proof: The lemma shows that the series expressing $f(z)$ and its apparent k^{th} derivative $\sum_n c_n \binom{n}{k} z^{n-k}$ all converge for $|z| < 1$. The usual direct proof of Abel's theorem on the differentiability of (scalar-valued) power series can be adapted to prove the infinite differentiability of the X-valued function given by this power series, as follows. Let

$$g(z) = \sum_{n \ge 0} n c_n\, z^{n-1}$$

Then

$$\frac{f(z)-f(w)}{z-w} - g(w) = \sum_{n\geq 1} c_n \left(\frac{z^n - w^n}{z-w} - nw^{n-1}\right)$$

For $n = 1$, the expression in the parentheses is 1. For $n > 1$, it is

$$\left(z^{n-1} + z^{n-2}w + \ldots + zw^{n-2} + w^{n-1}\right) - nw^{n-1}$$

$$= (z^{n-1} - w^{n-1}) + (z^{n-2}w - w^{n-1}) + \ldots + (z^2w^{n-3} - w^{n-1}) + (zw^{n-2} - w^{n-1}) + (w^{n-1} - w^{n-1})$$

$$= (z-w)\left[(z^{n-2} + \ldots + w^{n-2}) + w(z^{n-3} + \ldots + w^{n-3}) + \ldots + w^{n-3}(z+w) + w^{n-2} + 0\right]$$

$$= (z-w)\sum_{k=0}^{n-2} (k+1)\, z^{n-2-k}\, w^k$$

For $|z| \leq r$ and $|w| \leq r$ the latter expression is dominated by

$$|z-w| \cdot r^{n-2}\, \frac{n(n-1)}{2} < |z-w| \cdot n^2\, r^{n-2}$$

Let U be a balanced neighborhood of 0 in X, and t a sufficiently large real number such that for all complex ζ with $|\zeta| \geq t$ all c_n lie in ζU. For $|z| \leq r < 1$ and $|w| \leq r < 1$, by the lemma,

$$\frac{f(z)-f(w)}{z-w} - g(w) = (z-w)\sum_{n\geq 2} c_n \cdot \left(\sum_{k=0}^{n-2}(k+1)\, z^{n-2-k}\, w^k\right) \in (z-w)\cdot\left(\sum_n n^2\, r^{n-2}\right)\cdot tU$$

Thus, for any given convex balanced neighborhood U of 0 in X, as $z \to w$

$$\frac{f(z)-f(w)}{z-w} - g(w)$$

eventually lies in U. ///

[3.0.3] Corollary: Let c_n be a sequence of vectors in a Banach space X such that for some $r > 0$ the series $\sum |c_n| \cdot r^n$ converges in X. Then for $|z| < r$ the series $f(z) = \sum c_n z^n$ converges and gives a holomorphic (infinitely-many times complex-differentiable) X-valued function. ///

[Grothendieck 1953] A. Grothendieck, *Sur certains espaces de fonctions holomorphes, I, II*, J. Reine Angew. Math. **192** (1953), 35–64 and 77–95.
