Distributions supported on hyperplanes

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[0.0.1] Theorem: A distribution $u$ on $\mathbb{R}^{m+n} \approx \mathbb{R}^m \times \mathbb{R}^n$ supported on $\mathbb{R}^m \times \{0\}$, is uniquely expressible as a locally finite sum of transverse differentiations followed by restriction and evaluations, namely, a locally finite sum

$u = \sum_{\alpha} u_{\alpha} \circ \text{res}_{\mathbb{R}^m \times \{0\}} \circ D^{\alpha}$

where $\alpha$ is summed over multi-indices $(\alpha_1, \ldots, \alpha_n)$, $D^{\alpha}$ is the corresponding differential operator on $\{0\} \times \mathbb{R}^n$, and $u_{\alpha}$ are distributions on $\mathbb{R}^m \times \{0\}$. Further,

$spt \ u_{\alpha} \times \{0\} \subset spt \ u$  \hspace{1cm} (for all multi-indices $\alpha$)

Proof: For brevity, let

$\rho = \text{res}_{\mathbb{R}^m \times \{0\}} : C_c^\infty (\mathbb{R}^m \times \mathbb{R}^n) \rightarrow C_c^\infty (\mathbb{R}^m)$

be the natural restriction map of test functions on $\mathbb{R}^m \times \mathbb{R}^n$ to $\mathbb{R}^m \times \{0\}$, by

$(\rho f)(x) = f(x,0) \hspace{1cm} \text{(for } x \in \mathbb{R}^m)$

The adjoint $\rho^* : D(\mathbb{R}^m) \rightarrow D(\mathbb{R}^{m+n})$ is a continuous map of distributions on $\mathbb{R}^m$ to distributions on $\mathbb{R}^m \times \mathbb{R}^n$, defined by

$(\rho^* u)(f) = u(\rho(f))$

First, if we could apply $u$ to functions of the form $F(x,y) = f(x) \cdot y^\alpha$, and if $u$ had an expression as a sum as in the statement of the theorem, then

$u(f(x) \cdot \frac{y^\alpha}{\alpha!}) = (-1)^{|\beta|} \cdot u_\beta(f) \cdot \beta!$

since most of the transverse derivatives evaluated at 0 vanish. This is not quite legitimate, since $y^\alpha$ is not a test function. However, we can take a test function $\psi$ on $\mathbb{R}^n$ that is identically 1 near 0, and consider $\psi(y) \cdot y^\alpha$ instead of $y^\alpha$, and reach the same conclusion.

Thus, if there exists such an expression for $u$, it is unique. Further, this computation suggests how to specify the $u_{\alpha}$s, namely,

$u_\beta(f) = u(f(x) \otimes \frac{y^\beta}{\beta!} \cdot \psi(y) \cdot (-1)^{|\beta|})$

This would also show the containment of the supports.

Show that the sum of these $u_\beta$’s does give $u$. Given an open $U$ in $\mathbb{R}^{m+n}$ with compact closure, $u$ on $\mathcal{D}_U$ has some finite order $k$. As a slight generalization of the fact that distributions supported on $\{0\}$ are finite linear combinations of Dirac delta and its derivatives, we have

[0.0.2] Lemma: Let $v$ be a distribution of finite order $k$ supported on a compact set $K$. For a test function $\varphi$ whose derivatives up through order $k$ vanish on $K$, $v(\varphi) = 0$.

For any test function $F(x,y)$,

$\Phi(x,y) = F(x,y) - \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \frac{y^\alpha}{\alpha!} \psi(y) (D^\alpha F)(x,0)$

has all derivatives vanishing to order $k$ on the closure of $U$. Thus, by the lemma, $u(\Phi) = 0$, which proves that $u$ is equal to that sum, and also proves the local finiteness.

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