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# *Paley-Wiener theorems*

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Of course, the original version [Paley-Wiener 1934] referred to $L^2$ functions, not distributions. The distributional aspect is from [Schwartz 1952]. We emphasize Gelfand-Pettis vector-valued integral techniques. Proofs are given just for $\mathbb{R}$, where all ideas are already manifest.

**[0.0.1] Theorem:** A test function $f$ supported on a closed ball $B_r$ of radius $r$ at the origin in $\mathbb{R}$ has Fourier transform $\widehat{f}$ extending to an entire function on $\mathbb{C}$, with

$$|\widehat{f}(z)| \ll_N (1+|z|)^{-N} e^{r\cdot|y|} \qquad \text{(for } z = x+iy \in \mathbb{C}, \text{ for every } N\text{)}$$

Conversely, an entire function satisfying such an estimate has (inverse) Fourier transform which is a test function supported in the ball of radius $r$.

**[0.0.2] Remark:** Most of the proof is expected. The interesting point is that rate-of-growth in the imaginary part determines the support of the inverse Fourier transforms.

*Proof:* First, the integral for $\widehat{f}(z)$ is the integral of the compactly-supported, continuous, entire-function-valued [1] function,

$$\xi \longrightarrow \Big(z \to f(\xi)\cdot e^{-i\xi z}\Big)$$

Thus, the Gelfand-Pettis integral exists, and is entire. Multiplication by $z$ is converted to differentiation inside the integral,

$$(-iz)^N \cdot \widehat{f}(z) \;=\; \int_{B_r} \frac{\partial^N}{\partial\xi^N} e^{-iz\cdot\xi}\cdot f(\xi)\,d\xi \;=\; (-1)^N \int_{B_r} e^{-iz\cdot\xi}\cdot \frac{\partial^N}{\partial\xi^N} f(\xi)\,d\xi$$

by integration by parts. Note that differentiation does not enlarge support. Thus,

$$|\widehat{f}(z)| \ll_N (1+|z|)^{-N}\cdot \Big|\int_{B_r} e^{-iz\cdot\xi} f^{(N)}(\xi)\,d\xi\Big| \;\le\; (1+|z|)^{-N}\cdot e^{r\cdot|y|}\cdot \Big|\int_{B_r} e^{-ix\cdot\xi} f^{(N)}(\xi)\,d\xi\Big|$$

$$\le\; (1+|z|)^{-N}\cdot e^{r\cdot|y|}\cdot \int_{B_r} |f^{(N)}(\xi)|\,d\xi \;\ll_{f,N}\; (1+|z|)^{-N}\cdot e^{r\cdot|y|}$$

Conversely, let $F$ be an entire function with the indicated growth and decay property, and show that

$$\varphi(\xi) \;=\; \int e^{ix\xi}\,F(x)\,dx$$

is a test function with support inside $B_r$. Note that the assumptions on $F$ do *not* directly assert that $F$ is Schwartz, so we cannot directly conclude that $\varphi$ is smooth. Nevertheless, a similar obvious computation would give

$$\int (ix)^N\cdot e^{ix\xi}\,F(x)\,dx \;=\; \int \frac{\partial^N}{\partial\xi^N} e^{ix\xi}\,F(x)\,dx \;=\; \frac{\partial^N}{\partial\xi^N}\int e^{ix\xi}\,F(x)\,dx$$

Of course, moving the differentiation outside the integral is *necessary*. As expected, it is *justified* in terms of Gelfand-Pettis integrals, as follows. Since $F$ strongly vanishes at $\infty$, the integrand extends continuously to

[1] As usual, the space of entire functions is given the sups-on-compacts semi-norms $\sup_{z\in K}|f(z)|$. Since $\mathbb{C}$ can be covered by countably-many compacts, this topology is metrizable. Cauchy's integral formula proves *completeness*, so this space is Fréchet.

the stereographic-projection one-point compactification of $\mathbb{R}$, giving a compactly-supported smooth-function-valued function on this compactification. The measure on the compactification can be adjusted to be finite, taking advantage of the rapid decay of $F$:

$$\varphi(\xi) \;=\; \int e^{ix\xi}\, F(x)\, dx \;=\; \int e^{ix\xi}\, F(x)\,(1+x^2)^N \,\frac{dx}{(1+x^2)^N}$$

Thus, the Gelfand-Pettis integral exists, and $\varphi$ is smooth. Thus, in fact, the justification proves that such an integral of smooth functions is smooth without necessarily producing a formula for derivatives.

To see that $\varphi$ is supported inside $B_r$, observe that, taking $y$ of the same sign as $\xi$,

$$\Big|F(x+iy)\cdot e^{i\xi(x+iy)}\Big| \;\ll_N\; (1+|z|)^{-N}\cdot e^{(r-|\xi|)\cdot|y|}$$

Thus,

$$|\varphi(\xi)| \;\ll_N\; \int_{\mathbb{R}} (1+|z|)^{-N}\cdot e^{(r-|\xi|)\cdot|y|}\, dx \;\le\; e^{(r-|\xi|)\cdot|y|}\cdot \int_{\mathbb{R}} \frac{dx}{(1+|x|)^{-N}}$$

For $|\xi|>r$, letting $|y|\to+\infty$ shows that $\varphi(\xi)=0$. ///

**[0.0.3] Theorem:** The Fourier transform $\widehat{u}$ of a distribution $u$ supported in $B_r$, of order $N$, is (integration against) the function $x\to u(\xi\to e^{-ix\xi})$, which is *smooth*, and extends to an *entire* function satisfying

$$|\widehat{u}(z)| \;\ll\; (1+|z|)^N\cdot e^{r\cdot|y|}$$

Conversely, an entire function meeting such a bound is the Fourier transform of a distribution of order $N$ supported inside $B_r$.

*Proof:* Recall that the Fourier transform $\widehat{u}$ is the tempered distribution defined for Schwartz functions $\varphi$ by

$$\widehat{u}(\varphi) \;=\; u(\widehat{\varphi}) \;=\; u\Big(\xi\to\int_{\mathbb{R}} e^{-ix\xi}\,\varphi(x)\,dx\Big) \;=\; \int_{\mathbb{R}} u(\xi\to e^{-ix\xi})\,\varphi(x)\,dx$$

since $x\to(\xi\to e^{-ix\xi}\varphi(\xi)$ extends to a continuous smooth-function-valued function on the stereographic-projection one-point compactification of $\mathbb{R}$, and Gelfand-Pettis applies. Thus, as expected, $\widehat{u}$ is integration against $x\to u(\xi\to e^{-ix\xi})$.

The smooth-function-valued function $z\to(\xi\to e^{-iz\xi})$ is holomorphic in $z$. Compactly-supported distributions constitute the dual of $C^\infty(\mathbb{R})$, so application of $u$ gives a holomorphic *scalar*-valued function $z\to u(\xi\to e^{-iz\xi})$.

Let $\nu_N$ be the $N^{th}$-derivative seminorm on $C^\infty(B_r)$, so

$$|u(\varphi)| \;\ll_\varepsilon\; \nu_N(\varphi)$$

Then

$$|\widehat{u}(z)| \;=\; |u(\xi\to e^{-iz\xi})| \;\ll_\varepsilon\; \nu_N(\xi\to e^{-iz\xi}) \;\ll\; \sup_{B_r}\Big|(1+|z|)^N\, e^{-iz\xi}\Big| \;\le\; (1+|z|)^N e^{r\cdot|y|}$$

Conversely, let $F$ be an entire function with $|F(z)|\ll(1+|z|)^N e^{r\cdot|y|}$. Certainly $F$ is a tempered distribution, so $F=\widehat{u}$ for a tempered distribution. We show that $u$ is of order at most $N$ and has support in $B_r$.

With $\eta$ supported on $B_1$ with $\eta\ge 0$ and $\int\eta=1$, make an *approximate identity* $\eta_\varepsilon(x)=\eta(x/\varepsilon)/\varepsilon$ for $\varepsilon\to 0^+$. By the easy half of Paley-Wiener for test functions, $\widehat{\eta_\varepsilon}$ is entire and satisfies

$$|\widehat{\eta}_\varepsilon(z)| \;\ll_{\varepsilon,N}\; (1+|z|)^{-N}\cdot e^{\varepsilon\cdot|y|} \qquad \text{(for all } N)$$

Note that $\widehat{\eta}_\varepsilon(x) = \widehat{\eta}(\varepsilon \cdot x)$ goes to 1 as tempered distribution

By the more difficult half of Paley-Wiener for test functions, $F \cdot \widehat{\eta}_\varepsilon$ is $\widehat{\varphi}_\varepsilon$ for some test function $\varphi_\varepsilon$ supported in $B_{r+\varepsilon}$. Note that $F \cdot \widehat{\eta}_\varepsilon \to F$.

For Schwartz function $g$ with the support of $\widehat{g}$ not meeting $B_r$, $\widehat{g} \cdot \varphi_\varepsilon$ for sufficiently small $\varepsilon > 0$. Since $F \cdot \widehat{\eta}_\varepsilon$ is a Cauchy net as tempered distributions,

$$u(\widehat{g}) \;=\; \widehat{u}(g) \;=\; \int F \cdot g \;=\; \int \lim_\varepsilon (F \cdot \widehat{\eta}_\varepsilon)\, g \;=\; \lim_\varepsilon \int (F \cdot \widehat{\eta}_\varepsilon)\, g \;=\; \lim_\varepsilon \int \widehat{\varphi}_\varepsilon\, g \;=\; \lim_\varepsilon \int \varphi_\varepsilon\, \widehat{g} \;=\; 0$$

This shows that the support of $u$ is inside $B_r$. ///

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