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Introduction to Levi-Sobolev spaces

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The simplest case of a Levi-Sobolev *imbedding theorem* asserts that the +1-index Levi-Sobolev Hilbert space $H^1[a,b]$ described below is inside $C^o[a,b]$. This is a corollary of a Levi-Sobolev *inequality* asserting that the $C^o[a,b]$ norm is *dominated* by the $H^1[a,b]$ norm. All that is used is the fundamental theorem of calculus and the Cauchy-Schwarz-Bunyakowsky inequality. The point is that there is a large *Hilbert space* $H^1[a,b]$ inside the *Banach* space $C^o[a,b]$.

We will do much more with this idea subsequently.

We can think of $L^2[a,b]$ as

$$L^2[a,b] \;=\; \text{completion of } \; C^o[a,b] \text{ with respect to } |f|_{L^2} = \Big(\int_a^b |f(t)|^2\,dt\Big)^{1/2}$$

In fact, by this point we have shown that every $C^k[a,b]$ is dense in $L^2[a,b]$.

The +1-index *Levi-Sobolev* space[1] $H^1[a,b]$ is

$$H^1[a,b] \;=\; \text{completion of } \; C^1[a,b] \text{ with respect to } |f|_{H^1} = \Big(|f|^2_{L^2[a,b]} + |f'|^2_{L^2[a,b]}\Big)^{1/2}$$

[0.1] Theorem: *(Levi-Sobolev inequality)* On $C^1[a,b]$, the $H^1[a,b]$-norm *dominates* the $C^o[a,b]$-norm. That is, there is a constant C depending only on a,b such that $|f|_{C^o[a,b]} \le C \cdot |f|_{H^1[a,b]}$ for every $f \in C^1[a,b]$.

Proof: For $a \le x \le y \le b$, for $f \in C^1[a,b]$, the fundamental theorem of calculus gives

$$|f(y)-f(x)| \;=\; \Big|\int_x^y f'(t)\,dt\Big| \;\le\; \int_x^y |f'(t)|\,dt \;\le\; \Big(\int_x^y |f'(t)|^2\,dt\Big)^{1/2}\cdot\Big(\int_x^y 1\,dt\Big)^{1/2}$$

$$\le\; |f'|_{L^2}\cdot|x-y|^{\frac12} \;\le\; |f'|_{L^2}\cdot|a-b|^{\frac12}$$

Using the continuity of $f \in C^1[a,b]$, let $y \in [a,b]$ be such that $|f(y)| = \min_x |f(x)|$. Using the previous inequality,

$$|f(x)| \;\le\; |f(y)| + |f(x)-f(y)| \;\le\; \frac{\int_a^b |f(t)|\,dt}{|a-b|} \;+\; |f(x)-f(y)| \;\le\; \frac{\int_a^b |f|\cdot 1}{|a-b|} \;+\; |f'|_{L^2}\cdot|a-b|^{\frac12}$$

$$\le\; \frac{|f|_{L^2}^{\frac12}\cdot|a-b|^{\frac12}}{|a-b|} \;+\; |f'|_{L^2}\cdot|a-b|^{\frac12} \;=\; \frac{|f|_{L^2}^{\frac12}}{|a-b|^{\frac12}} \;+\; |f'|_{L^2}\cdot|a-b|^{\frac12} \;\le\; \Big(|f|_{L^2}+|f'|_{L^2}\Big)\cdot\Big(|a-b|^{-\frac12}+|a-b|^{\frac12}\Big)$$

$$\le\; 2(|f|^2+|f'|^2)^{1/2}\cdot\Big(|a-b|^{-\frac12}+|a-b|^{\frac12}\Big) \;=\; |f|_{H^1}\cdot 2\Big(|a-b|^{-\frac12}+|a-b|^{\frac12}\Big)$$

Thus, on $C^1[a,b]$ the H^1 norm dominates the C^o-norm. ///

[0.2] Corollary: *(Levi-Sobolev imbedding)* $H^1[a,b] \subset C^o[a,b]$.

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[1] ... also denoted $W^{1,2}[a,b]$, where the superscript 2 refers to L^2, rather than L^p. Beppo Levi noted the importance of taking Hilbert space completion with respect to this norm in 1906, giving a correct formulation of *Dirichlet's principle*. Sobolev's systematic development of these ideas was in the mid-1930's.

Proof: Since $H^1[a,b]$ is the H^1-norm completion of $C^1[a,b]$, every $f \in H^1[a,b]$ is an H^1-limit of functions $f_n \in C^1[a,b]$. That is, $|f - f_n|_{H^1[a,b]} \to 0$. Since the H^1-norm dominates the C^o-norm, $|f - f_n|_{C^o[a,b]} \to 0$. A C^o limit of continuous functions is continuous, so f is continuous. ///

In fact, we have a stronger conclusion than continuity, namely, a *Lipschitz condition* with exponent $\frac{1}{2}$:

[0.3] Corollary: (of proof of theorem) $|f(x) - f(y)| \le |f'|_{L^2} \cdot |x-y|^{\frac{1}{2}}$ for $f \in H^1[a,b]$. ///
