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Examples of function spaces

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We review natural topological vectorspaces of functions on relatively simple geometric objects, such as $\mathbb{R}$ or the circle $\mathbb{T}$.

In all cases, we specify a natural topology, in which differentiation or other natural operators are *continuous*, and so that the space is suitably *complete*.

Many familiar and useful spaces of continuous or differentiable functions, such as $C^k[a,b]$, have natural metric structures, and are *complete*. Often, the metric $d(,)$ comes from a *norm* $|\cdot|$, on the functions, giving Banach spaces.

Other natural function spaces, such as $C^\infty[a,b]$, $C^o(\mathbb{R})$, are *not* Banach, but still do have a metric topology and are complete: these are *Fréchet spaces*, appearing as (projective) *limits* of Banach spaces, as below. These lack some of the conveniences of Banach spaces, but their expressions as *limits* of Banach spaces is often sufficient.

Other important spaces, such as compactly-supported continuous functions $C^o_c(\mathbb{R})$ on $\mathbb{R}$, or compactly-supported smooth functions (test functions) $\mathcal{D}(\mathbb{R}) = C^\infty_c(\mathbb{R})$ on $\mathbb{R}$, are not metrizable so as to be *complete*. Nevertheless, some are expressible as *colimits* (sometimes called *inductive limits*) of Banach or Fréchet spaces, and such descriptions suffice for many applications. An *LF-space* is a countable ascending union of Fréchet spaces with each Fréchet subspace *closed* in the next. These are *strict colimits* or *strict inductive limits* of Fréchet spaces. These are generally *not* complete in the strongest sense, but, nevertheless, as demonstrated earlier, are *quasi-complete*, and this suffices for applications.

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1. *Non-Banach limits $C^k(\mathbb{R})$, $C^\infty(\mathbb{R})$ of Banach spaces $C^k[a,b]$*

For a *non-compact* topological space such as $\mathbb{R}$, the space $C^o(\mathbb{R})$ of continuous functions is *not* a Banach space with sup norm, because the sup of the absolute value of a continuous function may be $+\infty$.

But, $C^o(\mathbb{R})$ has a Fréchet-space structure: express $\mathbb{R}$ as a *countable union of compact subsets* $K_n = [-n, n]$. Despite the likely non-injectivity of the map $C^o(\mathbb{R}) \to C^o(K_i)$, giving $C^o(\mathbb{R})$ the (projective) limit topology $\lim_i C^o(K_i)$ is reasonable: certainly the restriction map $C^o(\mathbb{R}) \to C^o(K_i)$ should be continuous, as should all the restrictions $C^o(K_i) \to C^o(K_{i-1})$, whether or not these are *surjective*.

The argument in favor of giving $C^o(\mathbb{R})$ the limit topology is that a *compatible* family of maps $f_i : Z \to C^o(K_i)$ gives *compatible fragments* of functions F on $\mathbb{R}$. That is, for $z \in Z$, given $x \in \mathbb{R}$ take K_i such that x is in the interior of K_i. Then for all $j \geq i$ the function $x \to f_j(z)(x)$ is continuous near x, and the compatibility assures that all these functions are the same.

That is, the compatibility of these fragments is exactly the assertion that they fit together to make a function $x \to F_z(x)$ on the whole space X. Since continuity is a *local* property, $x \to F_z(x)$ is in $C^o(X)$. Further, there is *just one* way to piece the fragments together. Thus, diagrammatically,

$$\begin{tikzcd}
C^o(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^o(K_2) \arrow[r] & C^o(K_1) \\
& Z \arrow[ul, dotted, "z\to F_z"] \arrow[ur, dashed, "f_2"] \arrow[urr, dashed, "f_1"'] & &
\end{tikzcd}$$

Thus, $C^o(X) = \lim_n C^o(K_n)$ is a Fréchet space. Similarly, $C^k(\mathbb{R}) = \lim_n C^k(K_n)$ is a Fréchet space.

[1.1] Remark: The question of whether the restriction maps $C^o(K_n) \to C^o(K_{n-1})$ or $C^o(\mathbb{R}) \to C^o(K_n)$ are *surjective* need not be addressed.

Unsurprisingly, we have

[1.2] Theorem: $\frac{d}{dx} : C^k(\mathbb{R}) \to C^{k-1}(\mathbb{R})$ is continuous.

Proof: The argument is structurally similar to the argument for $\frac{d}{dx} : C^\infty[a,b] \to C^\infty[a,b]$. The differentiations $\frac{d}{dx} : C^k(K_n) \to C^{k-1}(K_n)$ are a compatible family, fitting into a commutative diagram

$$\begin{tikzcd}
C^{k-1}(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^{k-1}(K_{n+1}) \arrow[r] & C^{k-1}(K_n) \arrow[r] & \dots \\
C^k(\mathbb{R}) \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & C^k(K_{n+1}) \arrow[r] \arrow[u, "\frac{d}{dx}"] & C^k(K_n) \arrow[r] \arrow[u, "\frac{d}{dx}"] & \dots
\end{tikzcd}$$

Composing the projections with d/dx gives (dashed) induced maps from $C^k(\mathbb{R})$ to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in

$$\begin{tikzcd}
C^{k-1}(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^{k-1}(K_{n+1}) \arrow[r] & C^{k-1}(K_n) \arrow[r] & \dots \\
C^k(\mathbb{R}) \arrow[u, dotted, "\frac{d}{dx}"] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & C^k(K_{n+1}) \arrow[r] \arrow[u] & C^{k-1}(K_n) \arrow[r] \arrow[u] & \dots
\end{tikzcd}$$

That is, there is a unique continuous linear map $\frac{d}{dx} : C^k(\mathbb{R}) \to C^{k-1}(\mathbb{R})$ compatible with the differentiations on finite intervals. ///

Similarly,

[1.3] Theorem: $C^\infty(\mathbb{R}) = \lim_k C^k(\mathbb{R})$, also $C^\infty(\mathbb{R}) = \lim_n C^\infty(K_n)$, and $\frac{d}{dx} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is continuous.

Proof: From $C^\infty(\mathbb{R}) = \lim_k C^k(\mathbb{R})$ we can obtain the induced map d/dx, as follows. Starting with the commutative diagram

$$\begin{tikzcd}
C^\infty(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^{k-1}(\mathbb{R}) \arrow[r] & C^{k-1}(\mathbb{R}) \arrow[r] & \dots \\
C^\infty(\mathbb{R}) \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & C^k(\mathbb{R}) \arrow[r] \arrow[ur, "\frac{d}{dx}"] & C^k(\mathbb{R}) \arrow[r] \arrow[ur, "\frac{d}{dx}"] & \dots
\end{tikzcd}$$

Composing the projections with d/dx gives (dashed) induced maps from $C^k(\mathbb{R})$ to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in

$$\begin{tikzcd}
C^\infty(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \dots \arrow[r] & C^{k-1}(\mathbb{R}) \arrow[r] & C^{k-1}(\mathbb{R}) \arrow[r] & \dots \\
C^\infty(\mathbb{R}) \arrow[u, dotted, "\frac{d}{dx}"] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & C^k(\mathbb{R}) \arrow[r] \arrow[ur] & C^k(\mathbb{R}) \arrow[r] \arrow[ur] & \dots
\end{tikzcd}$$

A novelty is the assertion that (projective) limits *commute* with each other, so that the limits of $C^k(K_n)$ in k and in n can be taken in either order. Generally, in a situation

$$\begin{tikzcd}
 & \lim_j(\lim_i V_{ij}) \arrow[rr, dashed, bend left] \arrow[rrr, dashed, bend left] & \dots \arrow[r] & \lim_i V_{i2} \arrow[r] \arrow[ddd, bend left] \arrow[dddd, bend left] & \lim_i V_{i1} \arrow[ddd, bend left] \arrow[dddd, bend left] \\
\lim_i(\lim_j V_{ij}) \arrow[ddr, dashed, bend right] \arrow[dddr, dashed, bend right] & & & & \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
 & \lim_j V_{2j} \arrow[d] \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & V_{22} \arrow[r] \arrow[d] & V_{21} \arrow[d] \\
 & \lim_j V_{1j} \arrow[rr, bend right] \arrow[rrr, bend right] & \dots \arrow[r] & V_{12} \arrow[r] & V_{11}
\end{tikzcd}$$

the maps $\lim_j(\lim_i V_{ij}) \to V_{k\ell}$ induce a map $\lim_j(\lim_i V_{ij}) \to \lim_\ell V_{k\ell}$, which induce a unique $\lim_j(\lim_i V_{ij}) \to \lim_k(\lim_\ell V_{k\ell})$. Similarly, a unique map is induced in the opposite direction, and, for the usual reason, these are mutual inverses. ///

[1.4] Claim: For fixed $x \in \mathbb{R}$ and fixed non-negative integer k, the *evaluation map* $f \to f^{(k)}(x)$ is *continuous.*

Proof: Take n large enough so that $x \in [-n, n]$. Evaluation $f \to f^{(k)}(x)$ was shown to be continuous on $C^k[-n, n]$. Composing with the continuous $C^\infty(\mathbb{R}) \to C^k(\mathbb{R}) \to C^k[-n, n]$ gives the continuity. ///

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2. *Banach completion $C_o^k(\mathbb{R})$ of $C_c^k(\mathbb{R})$*

It is reasonable to ask about the completion of the space $C_c^o(\mathbb{R})$ of compactly-supported continuous functions in the metric given by the sup-norm, and, more generally, about the completion of the space $C_c^k(\mathbb{R})$ of compactly-supported k-times continuously differentiable functions in the metric given by the sum of the sups of the k derivatives.

The spaces $C_c^k(\mathbb{R})$ are *not complete* with those norms, because supports can *leak* out to infinity: for example, in fix any u such that $u(x) = 1$ for $|x| \le 1$, $0 \le u(x) \le 1$ for $1 \le |x| \le 2$, and $u(x) = 0$ for $|x| \ge 2$. Then

$$f(x) \;=\; \sum_{n=0}^{\infty} \frac{u(x-n)}{n^2}$$

converges in sup-norm, the partial sums have compact support, but the whole does not have compact support.

[2.1] Claim: The completion of the space $C_c^o(\mathbb{R})$ of compactly-supported continuous functions in the metric given by the sup-norm $|f|_{C^o} = \sup_{x\in\mathbb{R}} |f(x)|$ is the space $C_o^o(\mathbb{R})$ of continuous functions f *vanishing at infinity*, in the sense that, given $\varepsilon > 0$, there is a compact interval $K = [-N, N] \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$.

[2.2] Remark: Since we need to distinguish compactly-supported functions $C_c^o(\mathbb{R})$ from functions $C_o^o(\mathbb{R})$ going to 0 at infinity, we cannot use the latter notation for the former, unlike some sources.

Proof: This is almost a tautology. Given $f \in C_o^o(\mathbb{R})$, given $\varepsilon > 0$, let $K = [-N, N] \subset X$ be compact such that $|f(x)| < \varepsilon$ for $x \notin K$. It is easy to make an auxiliary function φ that is continuous, *compactly-supported*, real-valued function such that $\varphi = 1$ on K and $0 \le \varphi \le 1$ on X. Then $f - \varphi \cdot f$ is 0 on K, and of absolute value $|\varphi(x) \cdot f(x)| \le |f(x)| < \varepsilon$ off K. That is, $\sup_{\mathbb{R}} |f - \varphi \cdot f| < \varepsilon$, so $C_c^o(\mathbb{R})$ is dense in $C_o^o(\mathbb{R})$.

On the other hand, a sequence f_i in $C_c^o(\mathbb{R})$ that is a Cauchy sequence with respect to sup norm gives a Cauchy sequence in each $C^o[a, b]$, and converges uniformly pointwise to a continuous function on $[a, b]$ for every $[a, b]$. Let f be the pointwise limit. Given $\varepsilon > 0$ take i_o such that $\sup_x |f_i(x) - f_j(x)| < \varepsilon$ for all $i, j \ge i_o$. With K the support of f_{i_o},

$$\sup_{x\notin K} |f(x)| \;\le\; \sup_{x\notin K} |f(x) - f_{i_o}(x)| + \sup_{x\notin K} |f_{i_o}(x)| \;=\; \sup_{x\notin K} |f(x) - f_{i_o}(x)| + 0 \;\le\; \varepsilon \;<\; 2\varepsilon$$

showing that f goes to 0 at infinity. ///

[2.3] Corollary: Continuous functions vanishing at infinity are *uniformly* continuous.

Proof: For $f \in C_o^o(\mathbb{R})$, given $\varepsilon > 0$, let $g \in C_c^o(\mathbb{R})$ be such that $\sup |f - g| < \varepsilon$. By the uniform continuity of g, there is $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$, and

$$|f(x) - f(y)| \;\le\; |f(x) - g(x)| + |f(y) - g(y)| + |g(x) - g(y)| \;<\; 3\varepsilon$$

as desired. ///

The arguments for $C^k(\mathbb{R})$ are completely parallel: the completion of the space $C_c^k(\mathbb{R})$ of compactly supported k-times continuously differentiable functions is the space $C_c^k(\mathbb{R})$ of k-times continuously differentiable functions whose k derivatives go to zero at infinity. Similarly,

[2.4] Corollary: The space of C^k functions whose k derivatives all vanish at infinity have *uniformly* continuous derivatives. ///

[2.5] Claim: The limit $\lim_k C_o^k(\mathbb{R})$ is the space $C_o^\infty(\mathbb{R})$ of smooth functions all whose derivatives go to 0 at infinity. All those derivatives are *uniformly* continuous.

Proof: As with $C^\infty[a, b] = \bigcap_k C^k[a, b] = \lim_k C^k[a, b]$, by its very definition $C_o^\infty(\mathbb{R})$ is the intersection of the Banach spaces $C_o^k(\mathbb{R})$. For any compatible family $Z \to C_o^k(\mathbb{R})$, the compatibility implies that the image of Z is in that intersection. ///

[2.6] Corollary: The space $C_o^\infty(\mathbb{R})$ is a Fréchet space, so is *complete*.

Proof: As earlier, countable limits of Banach spaces are Fréchet. ///

[2.7] Remark: In contrast, the space of merely *bounded* continuous functions does not behave so well. Functions such as $f(x) = \sin(x^2)$ are not *uniformly* continuous. This has the bad side effect that

$\sup_x |f(x+h) - f(x)| = 1$ for all $h \neq 0$, which means that the *translation action* of $\mathbb{R}$ on that space of functions is *not continuous*.

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3. *Rapid-decay functions, Schwartz functions*

A continuous function f on $\mathbb{R}$ is *of rapid decay* when

$$\sup_{x\in\mathbb{R}} (1+x^2)^n \cdot |f(x)| \;<\; +\infty \qquad\qquad \text{(for every } n = 1, 2, \ldots)$$

With norm $\nu_n(f) = \sup_{x\in\mathbb{R}} (1+x^2)^n \cdot |f(x)|$, let the Banach space B_n be the completion of $C_c^o(\mathbb{R})$ with respect to the metric $\nu_n(f-g)$ associated to ν_n.

[3.1] Lemma: The Banach space B_n is isomorphic to $C_o^o(\mathbb{R})$ by the map $T : f \to (1+x^2)^n \cdot f$. Thus, B_n is the space of continuous functions f such that $(1+x^2)^n \cdot f(x)$ goes to 0 at infinity.

Proof: By design, $\nu_n(f)$ is the sup-norm of Tf. Thus, the result for $C_o^o(\mathbb{R})$ under sup-norm gives this lemma. ///

[3.2] Remark: Just as we want the completion $C_o^o(\mathbb{R})$ of $C_c^o(\mathbb{R})$, rather than the space of all *bounded* continuous functions, we want B_n rather than the space of all continuous functions f with $\sup_x (1+x^2) \cdot |f(x)| < \infty$. This distinction disappears in the limit, but it is only via the density of $C_c^o(\mathbb{R})$ in every B_n that it follows that $C_c^o(\mathbb{R})$ is dense in the space of continuous functions of rapid decay, in the corollary below.

[3.3] Claim: The space of continuous functions of rapid decay on $\mathbb{R}$ is the nested *intersection*, thereby the *limit*, of the Banach spaces B_n, so is Fréchet.

Proof: The key issue is to show that rapid-decay f is a ν_n-limit of *compactly-supported* continuous functions for every n. For each fixed n the function $f_n = (1+x^2)^n f$ is continuous and goes to 0 at infinity. From earlier, f_n is the sup-norm limit of *compactly supported* continuous functions F_{nj}. Then $(1+x^2)^{-n} F_{nj} \to f$ in the topology on B_n, and $f \in B_n$. Thus, the space of rapid-decay functions lies *inside* the intersection.

On the other hand, a function $f \in \bigcap_k B_k$ is continuous. For each n, since $(1+x^2)^{n+1}|f(x)|$ is continuous and goes to 0 at infinity, it has a finite sup σ, and

$$\sup_x (1+x^2)^n \cdot |f(x)| \;=\; \sup_x (1+x^2)^{-1} \cdot (1+x^2)^{n+1}|f(x)| \;\leq\; \sup_x (1+x^2)^{-1} \cdot \sigma \;<\; +\infty$$

This holds for all n, so f is of rapid decay. ///

[3.4] Corollary: The space $C_c^o(\mathbb{R})$ is *dense* in the space of continuous functions of rapid decay.

Proof: That every B_n is a completion of $C_c^o(\mathbb{R})$ is essential for this argument.

Use the model of the limit $X = \lim_n B_n$ as the diagonal in $\prod_n B_n$, with the product topology restricted to X. Let $p_n : \prod_k B_k \to B_n$ be the projection. Thus, given $x \in X$, there is a basis of neighborhood N of x in X of the form $N = X \cap U$ for an open U in the product of the form $U = \prod_n U_n$ with all but finitely-many $U_n = B_n$. Thus, for $y \in C_c^o(\mathbb{R})$ such that $p_n(y) \in p_n(N) = p_n(U)$ for the finitely-many indices such that $U_n \neq B_n$, we have $y \in N$. That is, approximating x in only *finitely-many* of the limitands B_n suffices to approximate x in the limit. Thus, density in the limitands B_n implies density in the limit. ///

[3.5] Remark: The previous argument applies generally, showing that a common subspace dense in all limitands is dense in the limit.

Certainly the operator of multiplication by $1+x^2$ preserves $C_c^o(\mathbb{R})$, and is a continuous map $B_n \to B_{n-1}$. Much as d/dx was treated earlier,

[3.6] Claim: Multiplication by $1+x^2$ is a continuous map of the space of continuous rapidly-decreasing functions to itself.

Proof: Let T denote the multiplication by $1+x^2$, and let $B = \lim_n B_n$ be the space of rapid-decay continuous functions. From the commutative diagram

$$\begin{tikzcd}
B \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots \arrow[r] & B_n \arrow[r] & B_{n-1} \arrow[r] & \ldots \\
B \arrow[rr, bend right] \arrow[rrr, bend right] & \ldots \arrow[r] & B_n \arrow[r] \arrow[ur, "T"] & B_{n-1} \arrow[r] \arrow[ur, "T"] & \ldots
\end{tikzcd}$$

composing the projections with T giving (dashed) induced maps from B to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in

$$\begin{tikzcd}
B \arrow[rr, bend left] \arrow[rrr, bend left] & \ldots \arrow[r] & B_n \arrow[r] & B_{n-1} \arrow[r] & \ldots \\
B \arrow[u, dotted, "T"] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[rr, bend right] \arrow[rrr, bend right] & \ldots \arrow[r] & B_n \arrow[r] \arrow[ur] & B_{n-1} \arrow[r] \arrow[ur] & \ldots
\end{tikzcd}$$

giving the *continuous* multiplication map on the space of rapid-decay continuous functions. ///

Similarly, adding differentiability conditions, the space of *rapidly decreasing* C^k functions is the space of k-times continuously differentiable functions f such that, for every $\ell = 0, 1, 2, \ldots, k$ and for every $n = 1, 2, \ldots$,

$$\sup_{x\in\mathbb{R}} (1+x^2)^n \cdot |f^{(\ell)}(x)| \;<\; +\infty$$

Let B_n^k be the completion of $C_c^k(\mathbb{R})$ with respect to the metric from the norm

$$\nu_n^k(f) \;=\; \sum_{0\le\ell\le k} \sup_{x\in\mathbb{R}} (1+x^2)^n |f^{(\ell)}(x)|$$

Essentially identical arguments give

[3.7] Claim: The space of C^k functions of rapid decay on $\mathbb{R}$ is the *nested intersection*, thereby the *limit*, of the Banach spaces B_n^k, so is Fréchet. ///

[3.8] Corollary: The space $C_c^k(\mathbb{R})$ is *dense* in the space of C^k functions of rapid decay. ///

Identifying B_n^k as a space of C^k functions with additional decay properties at infinity gives the obvious map $\frac{d}{dx} : B_n^k \to B_n^{k-1}$.

[3.9] Claim: $\frac{d}{dx} : B_n^k \to B_n^{k-1}$ is *continuous.*

Proof: Since B_n^k is the closure of $C_c^k(\mathbb{R})$, it suffices to check the continuity of $\frac{d}{dx} : C_c^k(\mathbb{R}) \to C_c^{k-1}(\mathbb{R})$ for the B_n^k and B_n^{k-1} topologies. As usual, that continuity was designed into the situation. ///

The space of *Schwartz functions* is

$$\mathscr{S}(\mathbb{R}) \;=\; \{\text{smooth functions } f \text{ all whose derivatives are of rapid decay}\}$$

One reasonable topology on $\mathscr{S}(\mathbb{R})$ is as a limit

$$\mathscr{S}(\mathbb{R}) \;=\; \bigcap_k \{C^k \text{ functions of rapid decay}\} \;=\; \lim_k \{C^k \text{ functions of rapid decay}\}$$

As a countable limit of Fréchet spaces, this makes $\mathscr{S}(\mathbb{R})$ Fréchet.

[3.10] Corollary: $\frac{d}{dx} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is *continuous*.

Proof: This is structurally the same as before: from the commutative diagram

$$\begin{tikzcd}
\mathscr{S}(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \cdots \arrow[r] & B_n^{k-1} \arrow[r] & B_n^{k-1} \arrow[r] & \cdots \\
\mathscr{S}(\mathbb{R}) \arrow[rr, bend right] \arrow[rrr, bend right] & \cdots \arrow[r] & B_n^{k} \arrow[u, "\frac{d}{dx}"] \arrow[r] & B_n^{k} \arrow[u, "\frac{d}{dx}"] \arrow[r] & \cdots
\end{tikzcd}$$

composing the projections with d/dx to give (dashed) induced maps from $\mathscr{S}(\mathbb{R})$ to the limitands, inducing a unique (dotted) continuous linear map to the limit:

$$\begin{tikzcd}
\mathscr{S}(\mathbb{R}) \arrow[rr, bend left] \arrow[rrr, bend left] & \cdots \arrow[r] & B_{n+1}^{k-1} \arrow[r] & B_n^{k-1} \arrow[r] & \cdots \\
\mathscr{S}(\mathbb{R}) \arrow[u, dotted, "\frac{d}{dx}"] \arrow[rru, dashed] \arrow[rrru, dashed] \arrow[rr, bend right] \arrow[rrr, bend right] & \cdots \arrow[r] & B_{n+1}^{k} \arrow[u] \arrow[r] & B_n^{k-1} \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

as desired. ///

Finally, to induce a canonical continuous map $T : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ by multiplication by $1 + x^2$, examine the behavior of this multiplication map on the auxiliary spaces B_n^k and its interaction with $\frac{d}{dx}$:

[3.11] Claim: $T : B_n^k \to B_{n-1}^{k-1}$ is continuous.

Proof: Of course,

$$\Big|\frac{d}{dx}\Big((1+x^2)\cdot f(x)\Big)\Big| \;=\; \Big|2x\cdot f(x) + (1+x^2)\cdot f'(x)\Big| \;\le\; 2\cdot(1+x^2)\cdot|f(x)| + (1+x^2)\cdot|f'(x)|$$

Thus, $T : C_c^k(\mathbb{R}) \to C_c^{k-1}(\mathbb{R})$ is continuous with the B_n^k and B_{n-1}^{k-1} topologies. As noted earlier, cofinal limits are isomorphic, so the same argument gives a unique continuous linear map $\mathscr{S}(\mathbb{R})$. ///

It is worth noting

[3.12] Claim: Compactly-supported smooth functions are *dense* in $\mathscr{S}$.

Proof: At least up to rearranging the order of limit-taking, the description of $\mathscr{S}$ above is as a limit of spaces in each of which compactly-supported smooth functions are dense. Thus, we claim a general result: for a limit $X = \lim_i X_i$ and compatible maps $f_i : V \to X_i$ with dense image, the induced map $f : V \to X$ has dense image. As earlier, the limit is the diagonal

$$D \;=\; \{\{x_i\} \in \prod_i X_i : x_i \to x_{i-1}, \text{ for all } i\} \;\;\subset\;\; \prod_i X_i$$

with the subspace topology from the product. Suppose we are given a finite collection of neighborhoods $x_{i_1} \in U_{i_1} \subset X_{i_1}, \ldots, x_{i_n} \in U_{i_n} \subset X_{i_n}$, with $x_{i_j} \to x_{i_k}$ if $i_j \ge i_k$. Take $i = \max_j i_j$, and U a neighborhood of x_i such that the image of U is inside every U_{i_j}, by continuity. Since the image of V is dense in X_i, there is $v \in V$ such that $f_i(v) \in U$. By compatibility, $f_{i_j}(v) \in U_{i_j}$ for all j. Thus, the image of V is dense in the limit. ///

4. *Non-Fréchet colimit* $C_c^\infty(\mathbb{R})$ *of Fréchet spaces*

The space of compactly-supported continuous functions

$$C_c^o(\mathbb{R}) \;=\; \text{compactly-supported continuous functions on } \mathbb{R}$$

is an *ascending union* of the subspaces

$$C^o_{[-n,n]} \;=\; \{f \in C^o(\mathbb{R}) \;:\; \text{spt} f \subset [-n,n]\}$$

Each space $C^o_{[-n,n]}$ is a Banach space, being a closed subspace of the Banach space $C^o[-n,n]$, further requiring vanishing of the functions on the boundary of $[-n,n]$. A closed subspace of a Banach space is a Banach space. Thus, $C_c^o(\mathbb{R})$ is an LF-space, and is *quasi-complete*.

Similarly,

$$C_c^k(\mathbb{R}) \;=\; \text{compactly-supported } C^k \text{ functions on } \mathbb{R}$$

is an *ascending union* of the subspaces

$$C^k_{[-n,n]} \;=\; \{f \in C^k(\mathbb{R}) \;:\; \text{spt} f \subset [-n,n]\}$$

Each space $C^k_{[-n,n]}$ is a Banach space, being a closed subspace of the Banach space $C^k[-n,n]$, further requiring vanishing of the functions and derivatives on the boundary of $[-n,n]$. A closed subspace of a Banach space is a Banach space. Thus, $C_c^k(\mathbb{R})$ is an LF-space, and is *quasi-complete*.

The space of *test functions* is

$$\mathcal{D}(\mathbb{R}) \;=\; C_c^\infty(\mathbb{R}) \;=\; \text{compactly-supported } C^\infty \text{ functions on } \mathbb{R}$$

is an *ascending union* of the subspaces

$$\mathcal{D}_{[-n,n]} \;=\; C^\infty_{[-n,n]} \;=\; \{f \in C^\infty(\mathbb{R}) \;:\; \text{spt} f \subset [-n,n]\}$$

Each space $\mathcal{D}_{[-n,n]}$ is a Fréchet space, being a closed subspace of the Fréchet space $C^\infty[-n,n]$, by further requiring vanishing of the functions and derivatives on the boundary of $[-n,n]$. A closed subspace of a Fréchet space is a Fréchet space. Thus, $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$ is an LF-space, and is *quasi-complete*.

The operator $\frac{d}{dx} : C^k[-n,n] \to C^{k-1}[-n,n]$ is continuous, and preserves the vanishing conditions at the endpoints, so restricts to a continuous map $\frac{d}{dx} : C^k_{[-n,n]} \to C^{k-1}_{[-n,n]}$ on the Banach sub-spaces of functions vanishing suitably at the endpoints. Composing with the inclusions $C^{k-1}_{[-n,n]} \to C_c^{k-1}(\mathbb{R})$ gives a compatible family of continuous maps $\frac{d}{dx} : C^k_{[-n,n]} \to C_c^{k-1}(\mathbb{R})$. This induces a unique continuous map on the colimit: $\frac{d}{dx} : C_c^k(\mathbb{R}) \to C_c^{k-1}(\mathbb{R})$.

Similarly, $\frac{d}{dx} : C^\infty[-n,n] \to C^\infty[-n,n]$ is continuous, and preserves the vanishing conditions at the endpoints, so restricts to a continuous map $\frac{d}{dx} : \mathcal{D}_{[-n,n]} \to \mathcal{D}_{[-n,n]}$ on the Frechet sub-spaces of functions vanishing to all orders at the endpoints. Composing with the inclusions $\mathcal{D}_{[-n,n]} \to \mathcal{D}(\mathbb{R})$ gives a compatible family of continuous maps $\frac{d}{dx} : \mathcal{D}_{[-n,n]} \to \mathcal{D}(\mathbb{R})$. This induces a unique continuous map on the colimit: $\frac{d}{dx} : \mathcal{D}(R) \to \mathcal{D}(\mathbb{R})$. Diagrammatically,

$$\begin{tikzcd}
\cdots \arrow[r] & C^\infty_{[-n,n]} \arrow[r] \arrow[d, "\frac{d}{dx}"] \arrow[rr, bend left] \arrow[drr, dashed, "\frac{d}{dx}"] & \cdots & \mathcal{D}(R) \arrow[d, dotted, "\frac{d}{dx}"] \\
\cdots \arrow[r] & C^\infty_{[-n,n]} \arrow[r] \arrow[rr, bend right] & \cdots & \mathcal{D}(\mathbb{R})
\end{tikzcd}$$

That is, $\frac{d}{dx}$ is continuous in the LF-space topology on test functions $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$.

[4.1] Claim: For fixed $x \in \mathbb{R}$ and non-negative integer k, the evaluation map $f \to f^{(k)}(x)$ on $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$ is *continuous.*

Proof: This evaluation map is continuous on $C^\infty[-n,n]$ for every large-enough n so that $x \in [-n,n]$, so is continuous on the closed subspace $\mathcal{D}_{[-n,n]}$ of $C^\infty[-n,n]$. The inclusions among these spaces are extend-by-0, so the evaluation map is the 0 map on $\mathcal{D}_{[-n,n]}$ if $|x| \geq n$. These maps to $\mathbb{C}$ fit together into a compatible family, so extend uniquely to a continuous linear map of the colimit $\mathcal{D}(\mathbb{R})$ to $\mathbb{C}$. ///

[4.2] Claim: For $F \in C^\infty(\mathbb{R})$, the map $f \to F \cdot f$ is a continuous map of $\mathcal{D}(\mathbb{R})$ to itself.

Proof: By the colimit characterization, it suffices to show that such a map is continuous on $C^\infty_{[-n,n]}$, or on the larger Fréchet space $C^\infty[-n,n]$ without vanishing conditions on the boundary. This is the limit of $C^k[-n,n]$, so it suffices to show that $f \to F \cdot f$ is a continuous map $C^k[-n,n] \to C^k[-n,n]$ for every k. The sum of sups of derivatives is

$$\sum_{0 \leq i \leq k} \sup_{|x| \leq n} \Big| \big(\frac{d}{dx}\big)^i (Ff)(x) \Big| \;\leq\; 2^k \Big(\sum_{0 \leq i \leq k} \sup_{|x| \leq n} |F^{(i)}(x)| \Big) \cdot \Big(\sum_{0 \leq i \leq k} \sup_{|x| \leq n} |f^{(i)}(x)| \Big)$$

Although F and its derivatives need not be bounded, this estimate only uses their boundedness on $[-n,n]$. This is a bad estimate, but sufficient for continuity. ///

[4.3] Claim: The inclusion $\mathcal{D}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is *continuous*, and the image is *dense.*

Proof: At least after changing order of limits, we have described $\mathscr{S}(\mathbb{R})$ as a limit of spaces in which $\mathcal{D}(\mathbb{R})$ is dense, so $\mathcal{D}(\mathbb{R})$ is dense in that limit.

The slightly more serious issue is that $\mathcal{D}(\mathbb{R})$ with its LF-space topology maps continuously to $\mathscr{S}(\mathbb{R})$. Since $\mathcal{D}(\mathbb{R})$ is a colimit, we need only check that the limitands (compatibly) map continuously. On a limitand $C^\infty_{[-n,n]}$, the norms

$$\nu_{N,k}(f) \;=\; \sup_x (1+x^2)^N \cdot |f^{(k)}(x)|$$

differ from the norms $\sup_x |f^{(k)}(x)|$ defining the topology on $C^\infty_{[-n,n]}$ merely by *constants*, namely, the sups of $(1+x^2)^N$ on $[-n,n]$. Thus, we have the desired continuity on the limitands. ///

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5. *LF-spaces of moderate-growth functions*

The space $C^o_{\mathrm{mod}}(\mathbb{R})$ of continuous functions of *moderate growth* on $\mathbb{R}$ is

$$C^o_{\mathrm{mod}}(\mathbb{R}) \;=\; \{f \in C^o(\mathbb{R}) \;:\; \sup_{x \in \mathbb{R}} (1+x^2)^{-N} \cdot |f(x)| \;<\; +\infty \text{ for some } N\}$$

Literally, it is an ascending union

$$C^o_{\mathrm{mod}}(\mathbb{R}) \;=\; \bigcup_N \Big\{ f \in C^o(\mathbb{R}) \;:\; \sup_{x \in \mathbb{R}} (1+x^2)^{-N} \cdot |f(x)| \;<\; +\infty \Big\}$$

However, it is ill-advised to use the individual spaces

$$B_N \;=\; \Big\{ f \in C^o(\mathbb{R}) \;:\; \sup_{x \in \mathbb{R}} (1+x^2)^{-N} \cdot |f(x)| \;<\; +\infty \Big\}$$

with norms $\nu_N(f) = \sup_{x\in\mathbb{R}} (1+x^2)^{-N} \cdot |f(x)|$ because $C^o_c(\mathbb{R})$ is not *dense* in these spaces B_N. Indeed, in the simple case $N = 0$, the norm ν_0 is the sup-norm, and the sup-norm closure of $C^o_c(\mathbb{R})$ is continuous functions *going to* 0 *at infinity*, which excludes many *bounded* continuous functions.

In particular, there are many bounded continuous functions f which are not *uniformly* continuous, and the translation action of $\mathbb{R}$ on such functions cannot be continuous: no matter how small $\delta > 0$ is, $\sup_{x\in\mathbb{R}} |f(x+\delta) - f(x)|$ may be large. For example, $f(x) = \sin(x^2)$ has this feature.

This difficulty does not mean that the characterization of the whole set of moderate-growth functions is incorrect, nor that the norms ν_N are inappropriate, but only that the Banach spaces B_N are too large, and that the topology of the whole should *not* be the strict colimit of the Banach spaces B_N. Instead, take the smaller

$$V_N \;=\; \text{completion of } C^o_c(\mathbb{R}) \text{ with respect to } \nu_N$$

As in the case of completion of $C^o_c(\mathbb{R})$ with respect to sup-norm ν_0,

[5.1] Claim: $V_N = \{\text{continuous } f \text{ such that } (1+x^2)^{-N} f(x) \text{ goes to 0 at infinity}\}$. ///

Of course, if $(1+x^2)^{-N} f(x)$ is merely *bounded*, then $(1+x^2)^{-(N+1)} f(x)$ then goes to 0 at infinity. Thus, as *sets*, $B_N \subset V_{N+1}$, but this inclusion *cannot be continuous*, since $C^o_c(\mathbb{R})$ is dense in V_{N+1}, but not in B_N. That is, there is a non-trivial effect on the topology in setting

$$C^o_{\mathrm{mod}} \;=\; \mathrm{colim}_N V_N$$

instead of the colimit of the too-large spaces B_N.

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6. *Strong operator topology*

For X and Y Hilbert spaces, the topology on continuous linear maps $\mathrm{Hom}^o(X, Y)$ given by seminorms

$$p_{x,U}(T) \;=\; \inf\{t > 0 : Tx \in tU\} \qquad (\text{for } T \in \mathrm{Hom}^o(X, Y))$$

where $x \in X$ and U is a convex, balanced neighborhood of 0 in Y, is the *strong operator topology*. Indeed, every neighborhood of 0 in Y contains an open *ball*, so this topology can also be given by seminorms

$$q_x(T) \;=\; |Tx|_Y \qquad (\text{for } T \in \mathrm{Hom}^o(X, Y))$$

where $x \in X$. The strong operator topology is weaker than the *uniform* topology given by the operator norm $|T| = \sup_{|x|\le 1} |Tx|_Y$.

The *uniform* operator-norm topology makes the space of operators a Banach space, certainly simpler than the strong operator topology, but the uniform topology is too strong for many purposes.

For example, *group actions* on Hilbert spaces are rarely continuous for the uniform topology: letting $\mathbb{R}$ act on $L^2(\mathbb{R})$ by $T_x f(y) = f(x+y)$, no matter how small $\delta > 0$ is, there is an L^2 function f with $|f|_{L^2} = 1$ such that $|T_\delta f - f|_{L^2} = \sqrt{2}$.

Despite the strong operator topology being less elementary than the uniform topology, the theorem on quasi-completeness shows that $\mathrm{Hom}^o(X, Y)$ with the strong operator topology is *quasi-complete*.

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7. *Generalized functions (distributions) on* $\mathbb{R}$

The most immediate definition of the space of *distributions* or *generalized functions* on $\mathbb{R}$ is as the dual $\mathcal{D}^* = \mathcal{D}(\mathbb{R})^* = C^\infty_c(\mathbb{R})^*$ to the space $\mathcal{D} = \mathcal{D}(\mathbb{R})$ of test functions, with the *weak dual topology* by seminorms $\nu_f(u) = |u(f)|$ for test functions f and distributions u.

Similarly, the *tempered* distributions are the weak dual $\mathscr{S}^* = \mathscr{S}(\mathbb{R})^*$, and the *compactly-supported* distributions are the weak dual $C^{\infty *} = C^\infty(\mathbb{R})^*$, in this context writing $C^\infty(\mathbb{R}) = C^\infty(\mathbb{R})$. Naming $C^{\infty *}$ *compactly-supported* will be justified below.

By dualizing, the continuous containments $\mathcal{D} \subset \mathscr{S} \subset C^\infty$ give continuous maps $C^{\infty *} \to \mathscr{S}^* \to \mathcal{D}^*$. When we know that $\mathcal{D}$ is *dense* in $\mathscr{S}$ and in C^∞, it will follow that these are *injections*. The most straightforward argument for density uses Gelfand-Pettis integrals, as in [???]. Thus, for the moment, we cannot claim that $C^{\infty *}$ and $\mathscr{S}^*$ *are* distributions, but only that they naturally *map to* distributions.

The general result on quasi-completeness of $\mathrm{Hom}^o(X,Y)$ for X an LF-space and Y quasi-complete shows that $\mathcal{D}^*$, $\mathscr{S}^*$, and $C^{\infty *}$ are *quasi-complete*, despite not being complete in the strongest possible sense.

The description of the space of distributions as the weak dual to the space of test functions falls far short of explaining its utility. There is a natural imbedding $\mathcal{D}(\mathbb{R}) \to \mathcal{D}(\mathbb{R})^*$ of test functions into distributions, by

$$f \to u_f \qquad \text{by} \qquad u_f(g) = \int_{\mathbb{R}} f(x)\, g(x)\, dx \qquad (\text{for } f, g \in \mathcal{D}(\mathbb{R}))$$

That is, via this imbedding we consider distributions to be *generalized functions*. Indeed, test functions $\mathcal{D}(\mathbb{R})$ are *dense* in $\mathcal{D}(\mathbb{R})^*$.

The simplest example of a distribution not obtained by integration against a test function on $\mathbb{R}$ is the *Dirac delta*, the evaluation map $\delta(f) = f(0)$, *continuous* for the LF-space topology on test functions.

This imbedding, and integration by parts, explain how to define $\frac{d}{dx}$ on distributions in a form compatible with the imbedding $\mathcal{D} \subset \mathcal{D}^*$: noting the sign, due to integration by parts,

$$\Big(\frac{d}{dx}u\Big)(f) \;=\; -\,u\Big(\frac{d}{dx}f\Big) \qquad (\text{for } u \in \mathcal{D}^* \text{ and } f \in \mathcal{D})$$

[7.1] Claim: $\frac{d}{dx} : \mathcal{D}^* \to \mathcal{D}^*$ is continuous.

Proof: By the nature of the weak dual topology, it suffices to show that for each $f \in \mathcal{D}$ and $\varepsilon > 0$ there are $g \in \mathcal{D}$ and $\delta > 0$ such that $|u(g)| < \delta$ implies $|(\frac{d}{dx}u)(f)| < \varepsilon$. Taking $g = \frac{d}{dx}f$ and $\delta = \varepsilon$ succeeds. ///

From [???], multiplications by $F \in C^\infty(\mathbb{R})$ give continuous maps $\mathcal{D}$ to itself. These multiplications are compatible with the imbedding $\mathcal{D} \to \mathcal{D}^*$ in the sense that

$$\int_{\mathbb{R}} (F\cdot u)(x)\; f(x)\; dx \;=\; \int_{\mathbb{R}} u(x)\; (F\cdot f)(x)\; dx \qquad (\text{for } F \in C^\infty(\mathbb{R}) \text{ and } u, f \in \mathcal{D}(\mathbb{R}))$$

Extend this to a map $\mathcal{D}^* \to \mathcal{D}^*$ by

$$(F\cdot u)(f) \;=\; u(F\cdot f) \qquad (\text{for } F \in C^\infty,\ u \in \mathcal{D}^*, \text{ and } f \in \mathcal{D})$$

[7.2] Claim: Multiplication operators $\mathcal{D}^* \to \mathcal{D}^*$ by $F \in C^\infty$ are continuous.

Proof: By the nature of the weak dual topology, it suffices to show that for each $f \in \mathcal{D}$ and $\varepsilon > 0$ there are $g \in \mathcal{D}$ and $\delta > 0$ such that $|u(g)| < \delta$ implies $|F\cdot u)(f)| < \varepsilon$. Taking $g = F\cdot f$ and $\delta = \varepsilon$ succeeds. ///

Since $\mathscr{S}$ is mapped to itself by Fourier transform [???], this gives a way to define Fourier transform on $\mathscr{S}^*$, as in [???].

Recall that the support of a *function* is the *closure* of the set on which it is non-zero, slightly complicating the notion of support for a *distribution* u: support of u is the *complement* of the *union* of all open sets U such that $u(f) = 0$ for all test functions f with support inside U.

[7.3] Theorem: A distribution with *support* $\{0\}$ is a finite linear combination of Dirac's δ and its derivatives.

Proof: Since $\mathcal{D}$ is a colimit of $\mathcal{D}_K$ over $K = [-n, n]$, it suffices to classify u in $\mathcal{D}_K^*$ with support $\{0\}$. We claim that a continuous linear functional on $\mathcal{D}_K = \lim_k C_K^k$ factors through some limitand

$$C_K^k \;=\; \{f \in C^k(K) :\; f^{(i)} \text{ vanishes on } \partial K \text{ for } 0 \le i \le k\}$$

This is a special case of

[7.4] Claim: Let $X = \lim_n B_n$ be a limit of Banach spaces, with the image of X *dense* in each B_n. A continuous linear map $T : \lim_n B_n \to Z$ from a, to a *normed* space Z factors through some limitand B_n. For $Z = \mathbb{C}$, the same conclusion holds without the density assumption.

Proof: Let $X = \lim_i B_i$ with projections $p_i : X \to B_i$. Each B_i is the closure of the image of X. By the continuity of T at 0, there is an open neighborhood U of 0 in X such that TU is inside the open unit ball at 0 in Z. By the description of the limit topology as the product topology restricted to the diagonal, there are finitely-many indices $i_1, \ldots, i_n$ and open neighborhoods V_{i_t} of 0 in B_{i_t} such that

$$\bigcap_{t=1}^{n} p_{i_t}^{-1}\big(p_{i_t} X \cap V_{i_t}\big) \;\subset\; U$$

We can make a *smaller* open in X by a condition involving a single limitand, as follows. Let j be *any* index with $j \ge i_t$ for all t, and

$$N \;=\; \bigcap_{t=1}^{n} p_{i_t,j}^{-1}\big(p_{i_t,j} B_j \cap V_{i_t}\big) \;\subset\; B_j$$

By the compatibility $p_{i_t}^{-1} = p_j^{-1} \circ p_{i_t,j}^{-1}$, we have $p_{i_t,j} N \subset V_{i_t}$ for $i_1, \ldots, i_n$, and $p_j^{-1}(p_j X \cap N) \subset U$. By the linearity of T, for any $\varepsilon > 0$,

$$T(\varepsilon \cdot p_j^{-1}\big(p_j X \cap N\big)) \;=\; \varepsilon \cdot T(p_j^{-1}\big(p_j X \cap N\big)) \;\subset\; \varepsilon\text{-ball in } Z$$

We claim that T factors through $p_j X$ with the subspace topology from B_j. One potential issue in general is that $p_j : X \to B_j$ can have a non-trivial kernel, and we must check that $\ker p_j \subset \ker T$. By the linearity of T,

$$T(\frac{1}{n} \cdot p_j^{-1}(p_j \cap N)) \;\subset\; \frac{1}{n}\text{-ball in } Z$$

so

$$T\left(\bigcap_n \frac{1}{n} \cdot p_j^{-1}(p_j X \cap N)\right) \;\subset\; \frac{1}{m}\text{-ball in } Z \qquad\qquad \text{(for all } m)$$

and then

$$T\left(\bigcap_n \frac{1}{n} \cdot p_j^{-1}(p_j \cap N)\right) \;\subset\; \bigcap_m \frac{1}{m}\text{-ball in } Z \;=\; \{0\}$$

Thus,

$$\bigcap_n p_j^{-1}(p_j X \cap \frac{1}{n} \cdot N) \;=\; \bigcap_n \frac{1}{n} \cdot p^{-1}(p_j X \cap N) \;\subset\; \ker T$$

Thus, for $x \in X$ with $p_j x = 0$, certainly $p_j x \in \frac{1}{n} N$ for all $n = 1, 2, \ldots$, and

$$x \;\in\; \bigcap_n p_j^{-1}(p_j X \cap \frac{1}{n} N) \;\subset\; \ker T$$

This proves the subordinate claim that T factors through $p_j : X \to B_j$ via a (not necessarily continuous) linear map $T' : p_j X \to Z$. The continuity follows from continuity at 0, which is

$$T(\varepsilon \cdot p_j^{-1}\big(p_j X \cap N\big)) \;=\; \varepsilon \cdot T(p_j^{-1}\big(p_j X \cap N\big)) \;\subset\; \varepsilon\text{-ball in } Z$$

Then $T' : p_j X \to Z$ extends to a map $B_j \to Z$ by continuity: given $\varepsilon > 0$, take symmetric convex neighborhood U of 0 in B_j such that $|T'y|_Z < \varepsilon$ for $y \in p_j X \cap U$. Let y_i be a Cauchy net in $p_j X$ approaching $b \in B_j$. For y_i and y_j inside $b + \frac{1}{2}U$, $|T'y_i - T'y_j| = |T'(y_i - y_j)| < \varepsilon$, since $y_i - y_j \in \frac{1}{2} \cdot 2U = U$. Then unambiguously define $T'b$ to be the Z-limit of the $T'y_i$. The closure of $p_j X$ in B_j is B_j, giving the desired map.

When u is a *functional*, that is, a map to $\mathbb{C}$, we can extend it by Hahn-Banach. ///

Returning to the proof of the theorem: thus, there is $k \geq 0$ such that u factors through a limitand C^k_K. In particular, u is continuous for the C^k topology on $\mathcal{D}_K$.

We need an auxiliary gadget. Fix a test function ψ identically 1 on a neighborhood of 0, bounded between 0 and 1, and (necessarily) identically 0 outside some (larger) neighborhood of 0. For $\varepsilon > 0$ let

$$\psi_\varepsilon(x) \;=\; \psi(\varepsilon^{-1}x)$$

Since the support of u is just $\{0\}$, for all $\varepsilon > 0$ and for all $f \in \mathcal{D}(\mathbb{R}^n)$ the support of $f - \psi_\varepsilon \cdot f$ does not include 0, so

$$u(\psi_\varepsilon \cdot f) \;=\; u(f)$$

Thus, for implied constant depending on k and K, but not on f,

$$|\psi_\varepsilon f|_k \;=\; \sup_{x\in K} \sum_{0\leq i\leq k} |(\psi_\varepsilon f)^{(i)}(x)| \;\ll\; \sum_{i\leq k}\sum_{0\leq j\leq i} \sup_x \varepsilon^{-j} \left|\psi^{(j)}(\varepsilon^{-1}x)\, f^{(i-j)}(x)\right|$$

For test function f vanishing to order k at 0, that is, $f^{(i)}(0) = 0$ for all $0 \leq i \leq k$, on a fixed neighborhood of 0, by a Taylor-Maclaurin expansion, $|f(x)| \ll |x|^{k+1}$, and, generally, for i^{th} derivatives with $0 \leq i \leq k$, $|f^{(i)}(x)| \ll |x|^{k+1-i}$. By design, all derivatives $\psi', \psi'', \ldots$ are identically 0 in a neighborhood of 0, so, for suitable implied constants independent of ε,

$$|\psi_\varepsilon f|_k \;\ll\; \sum_{0\leq i\leq k}\sum_{0\leq j\leq i} \varepsilon^{-j} \cdot \left|\psi^{(j)}(\varepsilon^{-1}x)\, f^{(i-j)}(x)\right| \;\ll\; \sum_{0\leq i\leq k}\sum_{j=0} \varepsilon^{-j}\cdot 1\cdot \varepsilon^{k+1-i}$$

$$=\; \sum_{0\leq i\leq k} \varepsilon^{k+1-i} \;\ll\; \varepsilon^{k+1-k} \;=\; \varepsilon$$

Thus, for sufficiently small $\varepsilon > 0$, for smooth f vanishing to order k at 0, $|u(f)| = |u(\psi_\varepsilon f)| \ll \varepsilon$, and $u(f) = 0$. That is,

$$\ker u \;\supset\; \bigcap_{0\leq i\leq k} \ker \delta^{(i)}$$

The conclusion, that u is a linear combination of the distributions $\delta, \delta', \delta^{(2)}, \ldots, \delta^{(k)}$, follows from

[7.5] Claim: A linear functional $\lambda \in V^*$ vanishing on the intersection $\bigcap_i \ker \lambda_i$ of kernels of a finite collection $\lambda_1, \ldots, \lambda_n \in V^*$ is a *linear combination* of the λ_i.

Proof: The linear map

$$q : V \;\longrightarrow\; \mathbb{C}^n \quad \text{by} \quad v \;\longrightarrow\; (\lambda_1 v, \,\ldots,\, \lambda_n v)$$

is *continuous* since each λ_i is continuous, and λ factors through q, as $\lambda = L \circ q$ for some linear functional L on $\mathbb{C}^n$. We know all the linear functionals on $\mathbb{C}^n$, namely, L is of the form

$$L(z_1, \ldots, z_n) \;=\; c_1 z_1 + \ldots + c_n z_n \qquad\qquad \text{(for some } c_i \in \mathbb{C})$$

Thus,

$$\lambda(v) \;=\; (L \circ q)(v) \;=\; L(\lambda_1 v, \,\ldots,\, \lambda_n v) \;=\; c_1\lambda_1(v) + \ldots + c_n\lambda_n(v)$$

expressing λ as a linear combination of the λ_i. ///

The *order* of a distribution $u : \mathcal{D} \to \mathbb{C}$ is the integer k, if such exists, such that u is continuous when $\mathcal{D}$ is given the weaker topology from $\mathrm{colim}_K C^k_K$. Not every distribution has finite order, but the claim [???] has a useful technical application:

[7.6] Corollary: A distribution $u \in \mathcal{D}^*$ with *compact support* has *finite order.*

Proof: Let ψ be a test function that is identically 1 on an open containing the support of u. Then

$$u(f) \;=\; u((1-\psi)\cdot f) + u(\psi \cdot f) \;=\; 0 + u(\psi \cdot f)$$

since $(1-\psi)\cdot f$ is a test function with support not meeting the support of u. With $K = \mathrm{spt}\,\psi$, this suggests that u factors through a subspace of $\mathcal{D}_K$ via $f \to \psi \cdot f \to u(\psi \cdot f)$, but there is the issue of continuity. Distinguishing things a little more carefully, the compatibility embodied in the commutative diagram

$$\begin{tikzcd} \ldots \arrow[r] & \mathcal{D}_K \arrow[r] \arrow[rr, bend left, "i"] \arrow[dr, "u_K"'] & \ldots & \mathcal{D} \arrow[dl, "u"] \\ & & \mathbb{C} & \end{tikzcd}$$

gives

$$u(f) \;=\; u(\psi \cdot f) \;=\; u\big(i(\psi f)\big) \;=\; u_K(\psi f)$$

The map u_K is continuous, as is the multiplication $f \to \psi f$. The map u_K is from the limit $\mathcal{D}_K$ of Banach spaces C^k_K to the normed space $\mathbb{C}$, so factors through some limitand C^k_K, by [???]. As in the proofs that multiplication is continuous in the C^∞ topology, by Leibniz' rule, the C^k norm of ψf is

$$|\psi f|_k \;=\; \sum_{0\le i\le k} \sup_{x\in K} |(\psi f)^{(i)}(x)| \;\ll\; \sum_{i\le k}\sum_{0\le j\le i} \sup_x \Big|\psi^{(j)}(x)\, f^{(i-j)}(x)\Big|$$

$$\ll\; \sum_{0\le i\le k} \sup_{x\in K} |f^{(i)}(x)| \cdot \sum_{j\le k} \sup_x |\psi^{(j)}(x)| \;=\; |f|_{C^k}\cdot|\psi|_{C^k}$$

Since ψ is fixed, this gives continuity in f in the C^k topology. ///

[7.7] Claim: In the inclusion $C^{\infty *} \subset \mathscr{S}^* \subset \mathcal{D}^*$, the image of $C^{\infty *}$ really is the collection of distributions with compact support.

Proof: On one hand the previous shows that $u \in \mathcal{D}^*$ with compact support can be composed as $u(f) = u_K(\psi f)$ for suitable $\psi \in \mathcal{D}$. The map $f \to \psi \cdot f$ is also continuous as a map $C^\infty \to \mathcal{D}$, so the same expression $f \to \psi f \to u_K(\psi f)$ *extends* $u \in \mathcal{D}^*$ to a continuous linear functional on C^∞.

On the other hand, let $u \in C^{\infty *}$. Composition of u with $\mathcal{D} \to C^\infty$ gives an element of $\mathcal{D}^*$, which we must check has compact support. By [???], C^∞ is a limit of the Banach spaces $C^k(K)$ with K running over compacts $[-n,n]$, *without* claiming that the image of C^∞ is necessarily dense in any of these. By [???], u factors through some limitand $C^k(K)$. The map $\mathcal{D} \to C^\infty$ is compatible with the *restriction* maps $\mathrm{Res}_K : \mathcal{D} \to C^k(K)$: the diagram

$$\begin{tikzcd} C^\infty \arrow[rr, bend left] & \ldots \arrow[r] & C^k(K) \arrow[r] \arrow[dr, "u"] & \ldots \\ & \mathcal{D} \arrow[ul] \arrow[ur, "\mathrm{Res}_K"'] & & \mathbb{C} \end{tikzcd}$$

commutes. For $f \in \mathcal{D}$ with support disjoint from K, $\mathrm{Res}_K(f) = 0$, and $u(f) = 0$. This proves that the support of the (induced) distribution is contained in K, so is compact. ///

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8. *Tempered distributions and Fourier transforms on* $\mathbb{R}$

One normalization of the *Fourier transform* integral is

$$\widehat{f}(\xi) \;=\; \mathbb{F}f(\xi) \;=\; \int_{\mathbb{R}} \overline{\psi}_\xi(x)\, f(x)\, dx \qquad\qquad (\text{with } \psi_\xi(x) = e^{2\pi i \xi x})$$

converges nicely for f in the space $\mathscr{S}(\mathbb{R})$ of Schwartz functions.

[8.1] Theorem: Fourier transform is a topological isomorphism of $\mathscr{S}(\mathbb{R})$ to itself, with Fourier inversion map $\varphi \to \check{\varphi}$ given by

$$\check{\varphi}(x) \;=\; \int_{\mathbb{R}} \psi_\xi(x)\; \widehat{f}(\xi)\; d\xi$$

Proof: Using the idea [14.3] that Schwartz functions extend to smooth functions on a suitable one-point compactification of $\mathbb{R}$ vanishing to infinite order at the point at infinity, Gelfand-Pettis integrals justify moving a differentiation under the integral,

$$\frac{d}{d\xi}\widehat{f}(\xi) \;=\; \frac{d}{d\xi}\int_{\mathbb{R}} \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; \int_{\mathbb{R}} \frac{\partial}{\partial \xi}\overline{\psi}_\xi(x)\, f(x)\, dx$$

$$=\; \int_{\mathbb{R}} (-2\pi i x)\, \overline{\psi}_\xi(x)\, f(x)\, dx \;=\; (-2\pi i)\int_{\mathbb{R}} \overline{\psi}_\xi(x)\, x f(x)\, dx \;=\; (-2\pi i)\widehat{xf}(\xi)$$

Similarly, with an integration by parts,

$$-2\pi i \xi \cdot \widehat{f}(\xi) \;=\; \int_{\mathbb{R}} \frac{\partial}{\partial x}\overline{\psi}_\xi(x)\cdot f(x)\, dx \;=\; -\mathbb{F}\frac{df}{dx}(\xi)$$

Thus, $\mathbb{F}$ maps $\mathscr{S}(\mathbb{R})$ to itself.

The natural idea to prove Fourier inversion for $\mathscr{S}(\mathbb{R})$, that unfortunately begs the question, is the obvious:

$$\int_{\mathbb{R}} \psi_\xi(x)\; \widehat{f}(\xi)\; d\xi \;=\; \int_{\mathbb{R}} \psi_\xi(x)\Big(\int_{\mathbb{R}} \overline{\psi}_\xi(t)\, f(t)\, dt\Big)\; d\xi \;=\; \int_{\mathbb{R}} f(t)\Big(\int_{\mathbb{R}} \psi_\xi(x-t)\, dt\Big)\; dt$$

If we could *justify* asserting that the inner integral is $\delta_x(t)$, which it *is*, then Fourier inversion follows. However, Fourier inversion for $\mathscr{S}(\mathbb{R})$ is used to make sense of that inner integral in the first place.

Despite that issue, a dummy *convergence factor* will legitimize the idea. For example, let $g(x) = e^{-\pi x^2}$ be the usual Gaussian. Various computations show that it is its own Fourier transform. For $\varepsilon > 0$, as $\varepsilon \to 0^+$, the dilated Gaussian $g_\varepsilon(x) = g(\varepsilon \cdot x)$ approaches 1 uniformly on compacts. Thus,

$$\int_{\mathbb{R}} \psi_\xi(x)\; \widehat{f}(\xi)\; d\xi \;=\; \int_{\mathbb{R}} \lim_{\varepsilon\to 0^+} g(\varepsilon\xi)\; \psi_\xi(x)\; \widehat{f}(\xi)\; d\xi \;=\; \lim_{\varepsilon\to 0^+}\int_{\mathbb{R}} g(\varepsilon\xi)\; \psi_\xi(x)\; \widehat{f}(\xi)\; d\xi$$

by *monotone convergence* or more elementary reasons. Then the iterated integral is legitimately rearranged:

$$\int_{\mathbb{R}} g(\varepsilon\xi)\; \psi_\xi(x)\; \widehat{f}(\xi)\; d\xi \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} g(\varepsilon\xi)\; \psi_\xi(x)\; \overline{\psi}_\xi(t)\, f(t)\; dt\; d\xi \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} g(\varepsilon\xi)\; \psi_\xi(x-t)\, f(t)\; d\xi\; dt$$

By changing variables in the definition of Fourier transform, $\widehat{g_\varepsilon} = \frac{1}{\varepsilon} g_{1/\varepsilon}$. Thus,

$$\int_{\mathbb{R}} \psi_\xi(x)\,\widehat{f}(\xi)\,d\xi \;=\; \int_{\mathbb{R}} \frac{1}{\varepsilon}\,g\big(\frac{x-t}{\varepsilon}\big)\,f(t)\,dt \;=\; \int_{\mathbb{R}} \frac{1}{\varepsilon}\,g\big(\frac{t}{\varepsilon}\big)\cdot f(x+t)\,dt$$

The sequence of function $g_{1/\varepsilon}/\varepsilon$ is not an *approximate identity* in the strictest sense, since the supports are the entire line. Nevertheless, the integral of each is 1, and as $\varepsilon \to 0^+$, the mass is concentrated on smaller and smaller neighborhoods of $0 \in \mathbb{R}$. Thus, for $f \in \mathscr{S}(\mathbb{R})$,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} \frac{1}{\varepsilon}\,g\big(\frac{t}{\varepsilon}\big)\cdot f(x+t)\,dt \;=\; f(x)$$

This proves Fourier inversion. In particular, this proves that Fourier transform *bijects* the Schwartz space to itself. ///

With Fourier inversion in hand, we can prove the Plancherel identity for Schwartz functions:

[8.2] Corollary: For $f, g \in \mathscr{S}$, the Fourier transform is an isometry in the $L^2(\mathbb{R})$ topology, that is, $\langle \widehat{f}, \widehat{g}\rangle = \langle f, g\rangle$.

Proof: There is an immediate preliminary identity:

$$\int_{\mathbb{R}} \widehat{f}(\xi)\,h(\xi)\,d\xi \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i\xi x}\,f(x)\,h(\xi)\,d\xi\,dx \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i\xi x}\,f(x)\,h(\xi)\,dx\,d\xi \;=\; \int_{\mathbb{R}} f(x)\,\widehat{h}(x)\,dx$$

To get from this identity to Plancherel requires, given $g \in \mathscr{S}$, existence of $h \in \mathscr{S}$ such that $\widehat{h} = \overline{g}$, with complex conjugation. By Fourier inversion on Schwartz functions, $h = (\overline{g})^{\vee}$ succeeds. ///

[8.3] Corollary: Fourier transform extends by continuity to an isometry $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Proof: Schwartz functions are dense in in $L^2(\mathbb{R})$. ///

[8.4] Corollary: Fourier transform extends to give a bijection of the space tempered distributions $\mathscr{S}^*$ to itself, by

$$\widehat{u}(\varphi) \;=\; u(\widehat{\varphi}) \qquad \text{(for all } \varphi \in \mathscr{S}\text{)}$$

Proof: Fourier transform is a topological isomorphism of $\mathscr{S}$ to itself. ///

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9. *Test functions and Paley-Wiener spaces*

Of course, the original [Paley-Wiener 1934] referred to L^2 functions, not distributions. The distributional aspect is from [Schwartz 1952]. An interesting point is that rate-of-growth of the Fourier transforms in the imaginary part determines the support of the inverse Fourier transforms.

The class PW of entire functions appearing in the following theorem is the *Paley-Wiener space* in one complex variable. The assertion is that, in contrast to the fact that Fourier transform maps the Schwartz space to itself, on test functions the Fourier transform has less symmetrical behavior, bijecting to the Paley-Wiener space.

[9.1] Theorem: A test function f supported on $[-r, r] \subset \mathbb{R}$ has Fourier transform $\widehat{f}$ extending to an entire function on $\mathbb{C}$, with

$$|\widehat{f}(z)| \;\ll_N\; (1+|z|)^{-N}\,e^{r\cdot|y|} \qquad \text{(for } z = x+iy \in \mathbb{C}\text{, for every } N\text{)}$$

Conversely, an entire function satisfying such an estimate has (inverse) Fourier transform which is a test function supported in $[-r, r]$.

Proof: First, the integral for $\widehat{f}(z)$ is the integral of the compactly-supported, continuous, entire-function-valued function,

$$\xi \longrightarrow \Big(z \to f(\xi) \cdot e^{-i\xi z}\Big)$$

where the space of entire functions is given the sups-on-compacts semi-norms $\sup_{z\in K} |f(z)|$. Since $\mathbb{C}$ can be covered by countably-many compacts, this topology is metrizable. Cauchy's integral formula proves *completeness*, so this space is Fréchet. Thus, the Gelfand-Pettis integral exists, and is entire. Multiplication by z is converted to differentiation inside the integral,

$$(-iz)^N \cdot \widehat{f}(z) \;=\; \int_{|\xi|\le r} \frac{\partial^N}{\partial \xi^N} e^{-iz\cdot\xi} \cdot f(\xi)\, d\xi \;=\; (-1)^N \int_{|\xi|\le r} e^{-iz\cdot\xi} \cdot \frac{\partial^N}{\partial \xi^N} f(\xi)\, d\xi$$

by integration by parts. Differentiation does not enlarge support, so

$$|\widehat{f}(z)| \;\ll_N\; (1+|z|)^{-N} \cdot \Big|\int_{|\xi|\le r} e^{-iz\cdot\xi}\, f^{(N)}(\xi)\, d\xi\Big| \;\le\; (1+|z|)^{-N} \cdot e^{r\cdot|y|} \cdot \Big|\int_{|\xi|\le r} e^{-ix\cdot\xi}\, f^{(N)}(\xi)\, d\xi\Big|$$

$$\le\; (1+|z|)^{-N} \cdot e^{r\cdot|y|} \cdot \int_{|\xi|\le r} |f^{(N)}(\xi)|\, d\xi \;\ll_{f,N}\; (1+|z|)^{-N} \cdot e^{r\cdot|y|}$$

Conversely, for an entire function F with the indicated growth and decay property, we show that

$$\varphi(\xi) \;=\; \int_{\mathbb{R}} e^{ix\xi}\, F(x)\, dx$$

is a test function with support inside $[-r, r]$. The assumptions on F do *not* directly include any assertion that F is Schwartz, so we cannot directly conclude that φ is smooth. Nevertheless, a similar obvious computation would give

$$\int_{\mathbb{R}} (ix)^N \cdot e^{ix\xi}\, F(x)\, dx \;=\; \int_{\mathbb{R}} \frac{\partial^N}{\partial \xi^N} e^{ix\xi}\, F(x)\, dx \;=\; \frac{\partial^N}{\partial \xi^N} \int_{\mathbb{R}} e^{ix\xi}\, F(x)\, dx$$

Moving the differentiation outside the integral is *necessary*, justified via Gelfand-Pettis integrals by a compactification device, as in [14.3], as follows. Since F strongly vanishes at ∞, the integrand extends continuously to the stereographic-projection one-point compactification of $\mathbb{R}$, giving a compactly-supported smooth-function-valued function on this compactification. The measure on the compactification can be adjusted to be finite, taking advantage of the rapid decay of F:

$$\varphi(\xi) \;=\; \int_{\mathbb{R}} e^{ix\xi}\, F(x)\, dx \;=\; \int_{\mathbb{R}} e^{ix\xi}\, F(x)\, (1+x^2)^N\, \frac{dx}{(1+x^2)^N}$$

Thus, the Gelfand-Pettis integral exists, and φ is smooth. Thus, in fact, the justification proves that such an integral of smooth functions is smooth without necessarily producing a formula for derivatives.

To see that φ is supported inside $[-r, r]$, observe that, taking y of the same sign as ξ,

$$\Big|F(x+iy) \cdot e^{i\xi(x+iy)}\Big| \;\ll_N\; (1+|z|)^{-N} \cdot e^{(r-|\xi|)\cdot|y|}$$

Thus,

$$|\varphi(\xi)| \;\ll_N\; \int_{\mathbb{R}} (1+|z|)^{-N} \cdot e^{(r-|\xi|)\cdot|y|}\, dx \;\le\; e^{(r-|\xi|)\cdot|y|} \cdot \int_{\mathbb{R}} \frac{dx}{(1+|x|)^{-N}}$$

For $|\xi| > r$, letting $|y| \to +\infty$ shows that $\varphi(\xi) = 0$. ///

[9.2] Corollary: We can topologize PW by requiring that the linear bijection $\mathcal{D} \to PW$ be a topological vector space isomorphism. ///

[9.3] Remark: The latter topology on PW is finer than the sups-on-compacts topology on all entire functions, since the latter cannot detect growth properties.

Let $\check{\varphi}(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi t}\,\varphi(\xi)\,d\xi$ be the inverse Fourier transform, mapping $PW \to \mathcal{D}$.

[9.4] Corollary: Fourier transform can be defined on *all* distributions $u \in \mathcal{D}^*$ by $\widehat{u}(\varphi) = u(\check{\varphi})$ for $\varphi \in PW$, giving an isomorphism $\mathcal{D}^* \to PW^*$ to the dual of the Paley-Wiener space. ///

For example, the exponential $t \to e^{iz\cdot t}$ with $z \in \mathbb{C}$ but $z \notin \mathbb{R}$ is *not* a tempered distribution, but is a distributions, and its Fourier transform is the Dirac delta $\delta_z \in PW'$.

Compactly-supported distributions have a similar characterization:

[9.5] Theorem: The Fourier transform $\widehat{u}$ of a distribution u supported in $[-r,r]$, of order N, is (integration against) the function $x \to u(\xi \to e^{-ix\xi})$, which is *smooth*, and extends to an *entire* function satisfying

$$|\widehat{u}(z)| \;\ll\; (1+|z|)^N \cdot e^{r\cdot|y|}$$

Conversely, an entire function meeting such a bound is the Fourier transform of a distribution of order N supported inside $[-r,r]$.

Proof: The Fourier transform $\widehat{u}$ is the tempered distribution defined for Schwartz functions φ by

$$\widehat{u}(\varphi) \;=\; u(\widehat{\varphi}) \;=\; u\Big(\xi \to \int_{\mathbb{R}} e^{-ix\xi}\,\varphi(x)\,dx\Big) \;=\; \int_{\mathbb{R}} u(\xi \to e^{-ix\xi})\,\varphi(x)\,dx$$

since $x \to (\xi \to e^{-ix\xi}\varphi(\xi)$ extends to a continuous smooth-function-valued function on the one-point compactification of $\mathbb{R}$, and Gelfand-Pettis applies. Thus, as expected, $\widehat{u}$ is integration against $x \to u(\xi \to e^{-ix\xi})$.

The smooth-function-valued function $z \to (\xi \to e^{-iz\xi})$ is holomorphic in z. Compactly-supported distributions constitute the dual of $C^\infty(\mathbb{R})$. Application of u gives a holomorphic *scalar*-valued function $z \to u(\xi \to e^{-iz\xi})$.

Let ν_N be the N^{th}-derivative seminorm on $C^\infty[-r,r]$, so

$$|u(\varphi)| \;\ll_\varepsilon\; \nu_N(\varphi)$$

Then

$$|\widehat{u}(z)| \;=\; |u(\xi \to e^{-iz\xi})| \;\ll_\varepsilon\; \nu_N(\xi \to e^{-iz\xi}) \;\ll\; \sup_{[-r,r]}\Big|(1+|z|)^N\,e^{-iz\xi}\Big| \;\le\; (1+|z|)^N e^{r\cdot|y|}$$

Conversely, let F be an entire function with $|F(z)| \ll (1+|z|)^N e^{r\cdot|y|}$. Certainly F is a tempered distribution, so $F = \widehat{u}$ for a tempered distribution. We show that u is of order at most N and has support in $[-r,r]$.

With η supported on $[-1,1]$ with $\eta \ge 0$ and $\int \eta = 1$, make an *approximate identity* $\eta_\varepsilon(x) = \eta(x/\varepsilon)/\varepsilon$ for $\varepsilon \to 0^+$. By the easy half of Paley-Wiener for test functions, $\widehat{\eta}_\varepsilon$ is entire and satisfies

$$|\widehat{\eta}_\varepsilon(z)| \;\ll_{\varepsilon,N}\; (1+|z|)^{-N} \cdot e^{\varepsilon\cdot|y|} \qquad \text{(for all } N\text{)}$$

Note that $\widehat{\eta}_\varepsilon(x) = \widehat{\eta}(\varepsilon\cdot x)$ goes to 1 as tempered distribution

By the more difficult half of Paley-Wiener for test functions, $F \cdot \widehat{\eta}_\varepsilon$ is $\widehat{\varphi}_\varepsilon$ for some test function φ_ε supported in $[-(r+\varepsilon), r+\varepsilon]$. Note that $F \cdot \widehat{\eta}_\varepsilon \to F$.

For Schwartz function g with the support of $\widehat{g}$ not meeting $[-r, r]$, $\widehat{g} \cdot \varphi_\varepsilon$ for sufficiently small $\varepsilon > 0$. Since $F \cdot \widehat{\eta}_\varepsilon$ is a Cauchy net as tempered distributions,

$$u(\widehat{g}) \;=\; \widehat{u}(g) \;=\; \int F \cdot g \;=\; \int \lim_\varepsilon (F \cdot \widehat{\eta}_\varepsilon)\; g \;=\; \lim_\varepsilon \int (F \cdot \widehat{\eta}_\varepsilon)\; g \;=\; \lim_\varepsilon \int \widehat{\varphi}_\varepsilon\; g \;=\; \lim_\varepsilon \int \varphi_\varepsilon\, \widehat{g} \;=\; 0$$

This shows that the support of u is inside $[-r, r]$. ///