(*April 18, 2017*)

Projectors, projection maps, orthogonal projections

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Now that we have some motivation to recall, or clarify, or further clarify, the notion of *projector* or *projection map*, we do so from scratch.

An element P in a ring with $P^2 = P$ is *idempotent*. Idempotency entails

$$P \cdot (1-P) \;=\; P^2 - P \;=\; 0$$

and

$$(1-P)^2 \;=\; 1 \cdot (1-P) - P \cdot (1-P) \;=\; 1 - P + 0 \;=\; 1 - P$$

so $1-P$ is also idempotent. The property $P_1 \cdot P_2 = 0$ for two idempotents P_1, P_2 is sometimes called *orthogonality*, but this is weaker than the notion of *orthogonal projection* in inner-product spaces.

Recall that the *orthogonal projection map* or *orthogonal projector* P of a Hilbert space V to a closed subspace $W \subset V$ is characterized by Pv being the unique point in W closest to v. [1] The image Pv is the orthogonal projection of v to W. The *idempotency* $P^2 = P$ follows, since $Pv \in W$ is itself the closest point in W to Pv.

[1.1] Claim: The orthogonal projection Pv of $v \in V$ to W is the unique element $v' \in W$ such that $v - v' \perp W$. [2]

Proof: The minimality characterization implies that for given $v \in V$, for all $0 \neq w \in W$, the non-degenerate quadratic polynomial function $f_w(t) = |v - Pv - tw|^2$ for $t \in \mathbb{R}$ takes its minimum at $t = 0$. Taking the derivative, this implies that $-\langle v - Pv, w\rangle - \langle w, v - Pv\rangle + 2t|w|^2$ vanishes at $t = 0$. That is, $-\langle v - Pv, w\rangle - \langle w, v - Pv\rangle = 0$ for all $w \in W$. Replacing w by μw with $\mu \in \mathbb{C}$ and $|\mu| = 1$ such that $\langle v - Pv, \mu w\rangle = |\langle v - Pv, w\rangle|$, this gives $\langle v - Pv, \mu w\rangle = 0$, and then $\langle v - Pv, w\rangle = 0$, so $v - Pv \perp W$.

Conversely, if $v' \in W$ with $\langle v - v', w\rangle = 0$ for all $w \in W$, then $v' = Pv$. Indeed, $f_w(t) = |v - v' - tw|^2 = |v|^2 + t^2|w|^2$, so the minimum occurs at $t = 0$, and v' is the closest point in W to v. ///

[1.2] Corollary: The kernel of the orthogonal projection P is exactly the orthogonal complement $W^\perp$.

Proof: On one hand, if $\langle v, w\rangle = 0$, then $\langle v - 0, w\rangle = 0$ for all $w \in W$, so $Pv = 0$, by the previous discussion. On the other hand, if $Pv = 0$, then $\langle v - 0, w\rangle = 0$ for all $w \in W$, and $v \in W^\perp$. ///

[1.3] Corollary: For P the orthogonal projector to W, the idempotent $1 - P$ is the orthogonal projector to $W^\perp$.

Proof: On one hand, $P(1-P)v = 0$ for all v, so $(1-P)v \in \ker P$, so $(1-P)v \in W^\perp$, by the previous corollary. That is, $1 - P$ maps to $W^\perp$, and

$$\langle v - (1-P)v, u\rangle \;=\; \langle Pv, u\rangle \;=\; 0 \qquad\qquad (\text{for all } u \in W^\perp)$$

[1] The existence and uniqueness of the point Pv follows from the *minimum principle*.

[2] This recharacterization facilitates proof that P is a *continuous linear* map. Given $x, y \in V$, $\langle (x+y) - (Px + Py), w\rangle = \langle x - Px, w\rangle + \langle y - Py, w\rangle = 0 + 0$ for all $w \in W$, so $P(x) + P(y) = P(x+y)$. Likewise for *linearity*: for scalar c, $\langle (cx) - cPx, w\rangle = c\langle x - Px, w\rangle = c \cdot 0$. For continuity, since $Pv \in W$ and $v - Pv \perp W$,

$$|v|^2 \;=\; |(v - Pv) + Pv|^2 \;=\; |v - Pv|^2 + \langle v - Pv, Pv\rangle + \langle Pv, v - Pv\rangle + |Pv|^2 \;=\; |v - Pv|^2 + |Pv|^2$$

which gives boundedness $|Pv| \leq |v|$, which gives continuity.

so by the *claim* $(1-P)v$ is the orthogonal projection of v to $W^\perp$. ///

[1.4] Example: Idempotency $T^2 = T$ of a linear map T does not guarantee orthogonality, as is already visible in two dimensions: in $\mathbb{C}^2$ with the usual hermitian inner product and standard basis e_1, e_2, define T by $Te_1 = e_1$ and $T(e_1+e_2) = 0$, so $Te_2 = -e_1$. Since there is a basis of eigenvectors with eigenvalues 0 or 1, T is idempotent. But T is not the orthogonal projection to $\mathbb{C}\cdot e_1$, because it sends e_2, which is orthogonal to $\mathbb{C}\cdot e_1$, to $-e_1$ rather than to 0.

[1.5] Claim: Idempotency and *self-adjointness* imply orthogonality. Conversely, idempotency and orthogonality imply *self-adjointness*.

Proof: Let $P^2 = P$ and $P^* = P$. For all $v, w \in V$,

$$\langle Pv, (1-P)w\rangle \;=\; \langle (1-P)^*Pv, w\rangle \;=\; \langle (1-P)Pv, w\rangle \;=\; \langle 0, w\rangle \;=\; 0$$

as claimed. Conversely, with P the orthogonal projector to W, to see that the adjoint P^* maps to W, take $x \in W^\perp$:

$$\langle x, P^*y\rangle \;=\; \langle Px, y\rangle \;=\; \langle 0, y\rangle \;=\; 0$$

since $\ker P = W^\perp$, from above. Then

$$\langle Px, y\rangle \;=\; \langle P^2x, y\rangle \;=\; \langle Px, P^*y\rangle \qquad\qquad (\text{for all } x, y \in V)$$

gives $\langle Px, y - P^*y\rangle = 0$ for all $x, y \in V$. Since P surjects to W, this implies that $y - P^*y \in W^\perp$. Then $Py = P^*y$, since Py is the unique element in W such that $y - Py \in W^\perp$. ///
