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Topological vector spaces

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1. *Banach spaces* $C^k[a,b]$

We give the vector space $C^k[a,b]$ of k-times continuously differentiable functions on an interval $[a,b]$ a metric which makes it *complete*. Mere *pointwise* limits of continuous functions easily fail to be continuous. First recall the standard

[1.1] Claim: The set $C^o(K)$ of complex-valued continuous functions on a compact set K is *complete* with the metric $|f-g|_{C^o}$, with the C^o-*norm* $|f|_{C^o} = \sup_{x\in K} |f(x)|$.

Proof: This is a typical three-epsilon argument. To show that a Cauchy sequence $\{f_i\}$ of continuous functions has a *pointwise* limit which is a continuous function, first argue that f_i has a pointwise limit at every $x \in K$. Given $\varepsilon > 0$, choose N large enough such that $|f_i - f_j| < \varepsilon$ for all $i, j \ge N$. Then $|f_i(x) - f_j(x)| < \varepsilon$ for any x in K. Thus, the sequence of values $f_i(x)$ is a Cauchy sequence of complex numbers, so has a limit $f(x)$. Further, given $\varepsilon' > 0$ choose $j \ge N$ sufficiently large such that $|f_j(x) - f(x)| < \varepsilon'$. For $i \ge N$

$$|f_i(x) - f(x)| \;\le\; |f_i(x) - f_j(x)| + |f_j(x) - f(x)| \;<\; \varepsilon + \varepsilon'$$

This is true for every positive ε', so $|f_i(x) - f(x)| \le \varepsilon$ for every x in K. That is, the pointwise limit is approached uniformly in $x \in [a,b]$.

To prove that $f(x)$ is continuous, for $\varepsilon > 0$, take N be large enough so that $|f_i - f_j| < \varepsilon$ for all $i, j \ge N$. From the previous paragraph $|f_i(x) - f(x)| \le \varepsilon$ for every x and for $i \ge N$. Fix $i \ge N$ and $x \in K$, and choose a small enough neigborhood U of x such that $|f_i(x) - f_i(y)| < \varepsilon$ for any y in U. Then

$$|f(x) - f(y)| \;\le\; |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f(y) - f_i(y)| \;\le\; \varepsilon + |f_i(x) - f_i(y)| + \varepsilon \;<\; \varepsilon + \varepsilon + \varepsilon$$

Thus, the pointwise limit f is continuous at every x in U. ///

Unsurprisingly, but significantly:

[1.2] Claim: For $x \in [a,b]$, the *evaluation* map $f \to f(x)$ is a continuous linear functional on $C^o[a,b]$.

Proof: For $|f-g|_{C^o} < \varepsilon$, we have

$$|f(x) - g(x)| \;\le\; |f-g|_{C^o} \;<\; \varepsilon$$

proving the continuity. ///

As usual, a real-valued or complex-valued function f on a closed interval $[a,b] \subset \mathbb{R}$ is *continuously differentiable* when it has a derivative which is itself a continuous function. That is, the limit

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists for all $x \in [a,b]$, and the function $f'(x)$ is in $C^o[a,b]$. Let $C^k[a,b]$ be the collection of k-times continuously differentiable functions on $[a,b]$, with the C^k-*norm*

$$|f|_{C^k} \;=\; \sum_{0\le i\le k} \sup_{x\in[a,b]} |f^{(i)}(x)| \;=\; \sum_{0\le i\le k} |f^{(i)}|_\infty$$

where $f^{(i)}$ is the i^{th} derivative of f. The *associated metric* on $C^k[a,b]$ is $|f-g|_{C^k}$.

Similar to the assertion about evaluation on $C^o[a,b]$,

[1.3] Claim: For $x \in [a,b]$ and $0 \le j \le k$, the *evaluation* map $f \to f^{(j)}(x)$ is a continuous linear functional on $C^k[a,b]$.

Proof: For $|f-g|_{C^k} < \varepsilon$,

$$|f^{(j)}(x) - g^{(j)}(x)| \;\le\; |f-g|_{C^k} \;<\; \varepsilon$$

proving the continuity. ///

We see that $C^k[a,b]$ is a Banach space:

[1.4] Theorem: The normed metric space $C^k[a,b]$ is complete.

Proof: For a Cauchy sequence $\{f_i\}$ in $C^k[a,b]$, all the pointwise limits $\lim_i f_i^{(j)}(x)$ of j-fold derivatives exist for $0 \le j \le k$, and are uniformly continuous. The issue is to show that $\lim_i f^{(j)}$ is differentiable, with derivative $\lim_i f^{(j+1)}$. It suffices to show that, for a Cauchy sequence f_n in $C^1[a,b]$, with pointwise limits $f(x) = \lim_n f_n(x)$ and $g(x) = \lim_n f_n'(x)$ we have $g = f'$. By the fundamental theorem of calculus, for any index i,

$$f_i(x) - f_i(a) \;=\; \int_a^x f_i'(t)\,dt$$

Since the f_i' *uniformly* approach g, given $\varepsilon > 0$ there is i_o such that $|f_i'(t) - g(t)| < \varepsilon$ for $i \ge i_o$ and for *all* t in the interval, so for such i

$$\Big| \int_a^x f_i'(t)\,dt - \int_a^x g(t)\,dt \Big| \;\le\; \int_a^x |f_i'(t) - g(t)|\;dt \;\le\; \varepsilon\cdot|x-a| \;\longrightarrow\; 0$$

Thus,

$$\lim_i f_i(x) - f_i(a) \;=\; \lim_i \int_a^x f_i'(t)\,dt \;=\; \int_a^x g(t)\,dt$$

from which $f' = g$. ///

By design, we have

[1.5] Theorem: The map $\frac{d}{dx} : C^k[a,b] \to C^{k-1}[a,b]$ is continuous.

Proof: As usual, for a linear map $T: V \to W$, by linearity $Tv - Tv' = T(v-v')$ it suffices to check continuity at 0. For Banach spaces the homogeneity $|\sigma\cdot v|_V = |\alpha|\cdot|v|_V$ shows that continuity is equivalent to existence of a constant B such that $|Tv|_W \le B\cdot|v|_V$ for $v \in V$. Then

$$|\frac{d}{dx}f|_{C^{k-1}} \;=\; \sum_{0\le i\le k-1} \sup_{x\in[a,b]} |(\frac{df}{dx})^{(i)}(x)| \;=\; \sum_{1\le i\le k} \sup_{x\in[a,b]} |f^{(i)}(x)| \;\le\; 1\cdot|f|_{C^k}$$

as desired. ///

2. *Non-Banach limit $C^\infty[a,b]$ of Banach spaces $C^k[a,b]$*

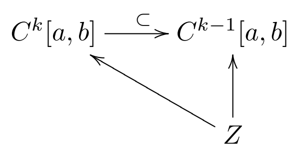
The space $C^\infty[a,b]$ of infinitely differentiable complex-valued functions on a (finite) interval $[a,b]$ in $\mathbb{R}$ is not a Banach space. [1] Nevertheless, the topology is *completely determined* by its relation to the Banach spaces $C^k[a,b]$. That is, there is a *unique* reasonable topology on $C^\infty[a,b]$. After explaining and proving this uniqueness, we also show that this topology is *complete metric*.

This function space can be presented as

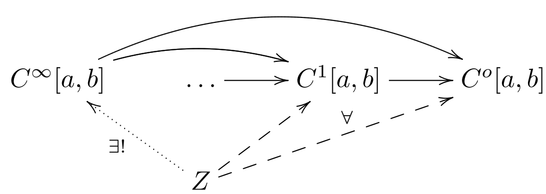
$$C^\infty[a,b] \;=\; \bigcap_{k\geq 0} C^k[a,b]$$

and we reasonably require that whatever topology $C^\infty[a,b]$ should have, each inclusion $C^\infty[a,b] \longrightarrow C^k[a,b]$ is continuous.

At the same time, given a family of *continuous linear* maps $Z \to C^k[a,b]$ from a vector space Z in some reasonable class (specified in the next section), with the *compatibility* condition of giving commutative diagrams



the image of Z actually lies in the intersection $C^\infty[a,b]$. Thus, diagrammatically, for every family of compatible maps $Z \to C^k[a,b]$, there is a *unique* $Z \to C^\infty[a,b]$ fitting into a commutative diagram



We require that this induced map $Z \to C^\infty[a,b]$ is *continuous.*

When we know that these conditions are met, we would say that $C^\infty[a,b]$ is the (projective) *limit* of the spaces $C^k[a,b]$, written

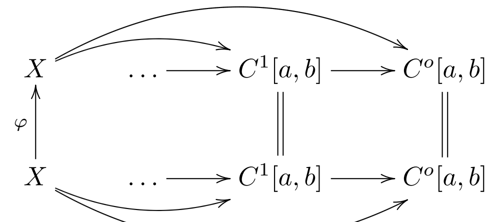
$$C^\infty[a,b] \;=\; \lim_k C^k[a,b]$$

with implicit reference to the inclusions $C^{k+1}[a,b] \to C^k[a,b]$ and $C^\infty[a,b] \to C^k[a,b]$.

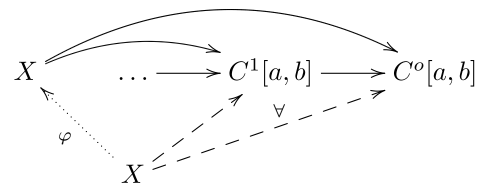
[2.1] Claim: Up to unique isomorphism, there exists at most one topology on $C^\infty[a,b]$ such that to every compatible family of continuous linear maps $Z \to C^k[a,b]$ from a topological vector space Z there is a unique continuous linear $Z \to C^\infty[a,b]$ fitting into a commutative diagram as just above.

Proof: Let X, Y be $C^\infty[a,b]$ with two topologies fitting into such diagrams, and show $X \approx Y$, and for a unique isomorphism. First, claim that the identity map $\mathrm{id}_X : X \to X$ is the only map $\varphi : X \to X$ fitting into a commutative diagram

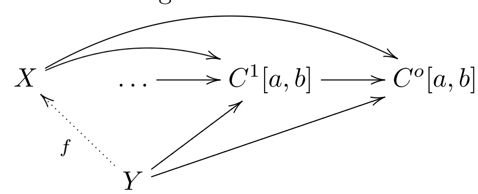
[1] It is not essential to prove that there is no reasonable Banach space structure on $C^\infty[a,b]$, but this can be readily proven in a suitable context.



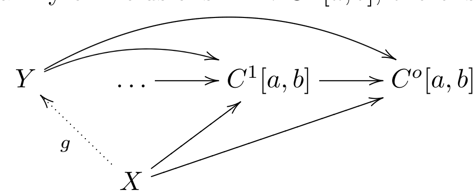
Indeed, given a compatible family of maps $X \to C^k[a,b]$, there is *unique* φ fitting into



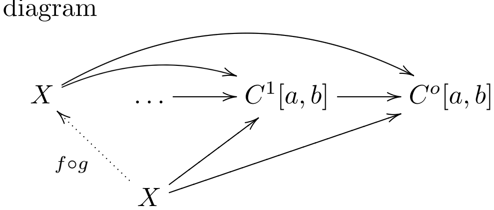
Since the identity map id_X fits, necessarily $\varphi = \mathrm{id}_X$. Similarly, given the compatible family of inclusions $Y \to C^k[a,b]$, there is unique $f : Y \to X$ fitting into



Similarly, given the compatible family of inclusions $X \to C^k[a,b]$, there is unique $g : X \to Y$ fitting into



Then $f \circ g : X \to X$ fits into a diagram

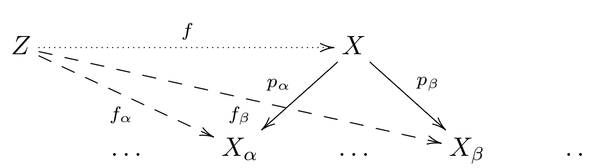


Therefore, $f \circ g = \mathrm{id}_X$. Similarly, $g \circ f = \mathrm{id}_Y$. That is, f, g are mutual inverses, so are isomorphisms of topological vector spaces. ///

Existence of a topology on $C^\infty[a,b]$ satisfying the condition above will be proven by identifying $C^\infty[a,b]$ as the obvious diagonal *closed subspace* of the *topological product* of the *limitands* $C^k[a,b]$:

$$C^\infty[a,b] \;=\; \{\{f_k : f_k \in C^k[a,b]\} : f_k = f_{k+1} \text{ for all } k\}$$

An arbitrary *product* of topological spaces X_α for α in an index set A is a topological space X with (*projections*) $p_\alpha : X \to X_\alpha$, such that every family $f_\alpha : Z \to X_\alpha$ of maps from any other topological space Z *factors through* the p_α *uniquely*, in the sense that there is a unique $f : Z \to X$ such that $f_\alpha = p_\alpha \circ f$ for all α. Pictorially, *all triangles commute* in the diagram



A similar argument to that for uniqueness of limits proves *uniqueness* of products up to unique isomorphism. *Construction* of products is by putting the usual product topology with basis consisting of products $\prod_\alpha Y_\alpha$ with $Y_\alpha = X_\alpha$ for all but finitely-many indices, on the Cartesian product of the *sets* X_α, whose existence we grant ourselves. Proof that this usual is *a* product amounts to unwinding the definitions. By uniqueness, in particular, despite the plausibility of the *box topology* on the product, it cannot function as a product topology since it differs from the standard product topology in general.

[2.2] Claim: Giving the diagonal copy of $C^\infty[a,b]$ inside $\prod_k C^k[a,b]$ the subspace topology yields a (projective) limit topology.

Proof: The projection maps $p_k : \prod_j C^j[a,b] \to C^k[a,b]$ from the whole product to the factors $C^k[a,b]$ are continuous, so their restrictions to the diagonally imbedded $C^\infty[a,b]$ are continuous. Further, letting $i_k : C^k[a,b] \to C^{k-1}[a,b]$ be the inclusion, on that diagonal copy of $C^\infty[a,b]$ we have $i_k \circ p_k = p_{k-1}$ as required.

On the other hand, *any* family of maps $\varphi_k : Z \to C^k[a,b]$ induces a map $\widetilde{\varphi} : Z \to \prod C^k[a,b]$ such that $p_k \circ \widetilde{\varphi} = \varphi_k$, by the property of the product. *Compatibility* $i_k \circ \varphi_k = \varphi_{k-1}$ implies that the image of $\widetilde{\varphi}$ is inside the diagonal, that is, inside the copy of $C^\infty[a,b]$. ///

A *countable* product of *metric* spaces X_k with metrics d_k has no canonical single metric, but is *metrizable*. One of many topologically equivalent metrics is the usual

$$d(\{x_k\},\{y_k\}) \;=\; \sum_{k=0}^\infty 2^{-k} \frac{d_k(x_k - y_k)}{d_k(x_k - y_k) + 1}$$

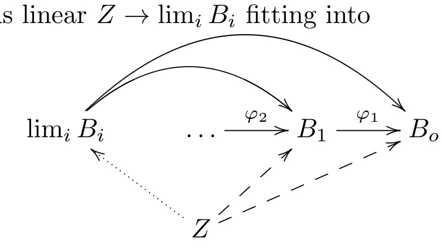
When the metric spaces X_k are *complete*, the product is complete. A closed subspace of a complete metrizable space is complete metrizable, so we have

[2.3] Corollary: $C^\infty[a,b]$ is complete metrizable. ///

Abstracting the above, for a (not necessarily countable) family

$$\ldots \xrightarrow{\varphi_2} B_1 \xrightarrow{\varphi_1} B_o$$

of Banach spaces with continuous linear transition maps as indicated, *not* recessarily requiring the continuous linear maps to be injective (or surjective), a *(projective) limit* $\lim_i B_i$ is a topological vector space with continuous linear maps $\lim_i B_i \to B_j$ such that, for every compatible family of continuous linear maps $Z \to B_i$ there is unique continuous linear $Z \to \lim_i B_i$ fitting into



The same *uniqueness* proof as above shows that there is at most one topological vector space $\lim_i B_i$. For *existence* by *construction*, the earlier argument needs only minor adjustment. The conclusion of complete metrizability would hold when the family is countable.

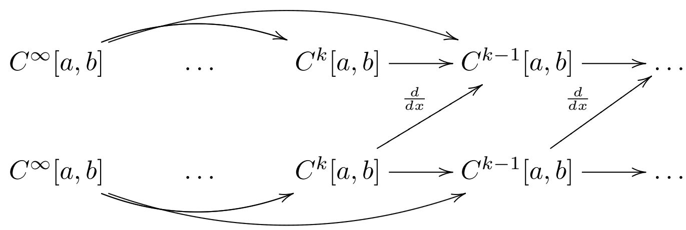
Before declaring $C^\infty[a,b]$ to be a *Fréchet* space, we must certify that it is *locally convex*, in the sense that every point has a local basis of *convex* opens. Normed spaces are immediately locally convex, because open balls are convex: for $0 \le t \le 1$ and x, y in the ε-ball at 0 in a normed space,

$$|tx + (1-t)y| \;\le\; |tx| + |(1-t)y| \;\le\; t|x| + (1-t)|y| \;<\; t\cdot\varepsilon + (1-t)\cdot\varepsilon \;=\; \varepsilon$$

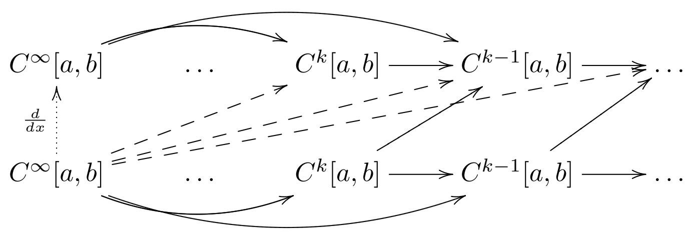
Product topologies of locally convex vectorspaces are locally convex, from the *construction* of the product. The construction of the limit as the diagonal in the product, with the subspace topology, shows that it is locally convex. In particular, *countable limits of Banach spaces are locally convex, hence, are Fréchet.* All spaces of practical interest are locally convex for simple reasons, so demonstrating local convexity is rarely interesting.

[2.4] Theorem: $\frac{d}{dx} : C^\infty[a,b] \to C^\infty[a,b]$ is continuous.

Proof: In fact, the differentiation operator is characterized via the expression of $C^\infty[a,b]$ as a limit. We already know that differentiation d/dx gives a continuous map $C^k[a,b] \to C^{k-1}[a,b]$. Differentiation is compatible with the inclusions among the $C^k[a,b]$. Thus, we have a commutative diagram



Composing the projections with d/dx gives (dashed) induced maps from $C^\infty[a,b]$ to the limitands, inducing a unique (dotted) continuous linear map to the limit, as in



This proves the continuity of differentiation in the limit topology. ///

In a slightly different vein, we have

[2.5] Claim: For all $x \in [a,b]$ and for all non-negative integers k, the evaluation map $f \to f^{(k)}(x)$ is a continuous linear map $C^\infty[a,b] \to \mathbb{C}$.

Proof: The inclusion $C^\infty[a,b] \to C^k[a,b]$ is continuous, and the evaluation of the k^{th} derivative is continuous. ///

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3. *Sufficient notion of topological vector space*

To describe a (projective) limit of Hilbert or Banach spaces by characterizing behavior in relation to *all* topological vectorspaces requires specification of what a topological vectorspace *should be.*

A *topological vector space* V (over $\mathbb{C}$) is a $\mathbb{C}$-vectorspace V with a topology on V in which *points are closed*, and so that *scalar multiplication*

$$x \times v \longrightarrow xv \qquad \text{(for } x \in k \text{ and } v \in V\text{)}$$

and *vector addition*

$$v \times w \longrightarrow v + w \qquad \text{(for } v, w \in V\text{)}$$

are *continuous*. For subsets X, Y of V, let

$$X + Y = \{x + y : x \in X, y \in Y\}$$

and

$$-X \;=\; \{-x : x \in X\}$$

The following trick is elementary, but indispensable. Given an open neighborhood U of 0 in a topological vectorspace V, continuity of vector addition yields an open neighborhood U' of 0 such that

$$U' + U' \;\subset\; U$$

Since $0 \in U'$, necessarily $U' \subset U$. This can be repeated to give, for any positive integer n, an open neighborhood U_n of 0 such that

$$\underbrace{U_n + \ldots + U_n}_{n} \;\subset\; U$$

In a similar vein, for fixed $v \in V$ the map $V \to V$ by $x \to x + v$ is a *homeomorphism*, being invertible by the obvious $x \to x - v$. Thus, *the open neighborhoods of v are of the form $v + U$ for open neighborhoods U of* 0. In particular, *a local basis at* 0 *gives the topology on a topological vectorspace.*

[3.1] Lemma: Given a compact subset K of a topological vectorspace V and a closed subset C of V not meeting K, there is an open neighborhood U of 0 in V such that

$$\text{closure}(K + U) \;\cap\; (C + U) \quad = \quad \phi$$

Proof: Since C is closed, for $x \in K$ there is a neighborhood U_x of 0 such that the neighborhood $x + U_x$ of x does not meet C. By continuity of vector addition

$$V \times V \times V \to V \quad \text{by} \quad v_1 \times v_2 \times v_3 \to v_1 + v_2 + v_3$$

there is a smaller open neighborhood N_x of 0 so that

$$N_x + N_x + N_x \;\subset\; U_x$$

By replacing N_x by $N_x \cap -N_x$, which is still an open neighborhood of 0, suppose that N_x is *symmetric* in the sense that $N_x = -N_x$.

Using this symmetry,

$$(x + N_x + N_x) \cap (C + N_x) \;=\; \phi$$

Since K is compact, there are finitely-many $x_1, \ldots, x_n$ such that

$$K \;\subset\; (x_1 + N_{x_1}) \cup \ldots \cup (x_n + N_{x_n})$$

Let $U = \bigcap_i \; N_{x_i}$. Since the intersection is finite, U is open. Then

$$K + U \;\subset\; \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + U) \;\subset\; \bigcup_{i=1,\ldots,n} (x_i + N_{x_i} + N_{x_i})$$

These sets do not meet $C + U$, by construction, since $U \subset N_{x_i}$ for all i. Finally, since $C + U$ is a union of opens $y + U$ for $y \in C$, it is open, so even the *closure* of $K + U$ does not meet $C + U$. ///

Conveniently, Hausdorff-ness of topological vectorspaces follows from the weaker assumption that points are closed:

[3.2] Corollary: A topological vectorspace is *Hausdorff.*

Proof: Take $K = \{x\}$ and $C = \{y\}$ in the lemma. ///

[3.3] Corollary: The topological closure $\bar{E}$ of a subset E of a topological vectorspace V can be expressed as

$$\bar{E} \;=\; \bigcap_U (E+U) \qquad\qquad \text{(where } U \text{ ranges over a local basis at 0)}$$

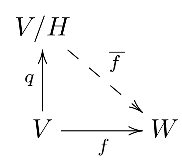
Proof: In the lemma, take $K = \{x\}$ and $C = \bar{E}$ for a point x of V not in C. Then we obtain an open neighborhood U of 0 so that $x+U$ does not meet $\bar{E}+U$. The latter contains $E+U$, so certainly $x \notin E+U$. That is, for x not in the closure, there is an open U containing 0 so that $x \notin E+U$. ///

As usual, for two topological vectorspaces V, W over $\mathbb{C}$, a function $f : V \longrightarrow W$ is *(k-)linear* when $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ for all $\alpha, \beta \in k$ and $x, y \in V$. Almost without exception we care about *continuous* linear maps, meaning linear maps continuous for the topologies on V, W. As expected, the *kernel* $\ker f$ of a linear map is

$$\ker f \;=\; \{v \in V : f(v) = 0\}$$

Being the inverse image of a closed set by a continuous map, the kernel is a *closed* subspace of V.

For a *closed* subspace H of a topological vectorspace V, the *quotient* V/H is *characterized* as topological vectorspace with linear quotient map $q : V \to V/H$ through which any continuous $f : V \to W$ with $\ker f \supset H$ *factors*, in the sense that there is a unique continuous linear $\overline{f} : V/H \to W$ giving a commutative diagram



Uniqueness of the quotient $q : V \to V/H$, up to unique isomorphism, follows by the usual categorical arguments, as with limits and products above. The *existence* of the quotient is proven by the usual construction of V/H as the collection of cosets $v + H$, with q given as usual by $q : v \longrightarrow v + H$. We verify that this construction succeeds in the proposition below.

The *quotient topology* on V/H is the *finest* topology such that the quotient map $q : V \to V/H$ is continuous, namely, a subset E of V/H is open if and only if $q^{-1}(E)$ is open.

For *non*-closed subspaces H, the quotient topology on the collection of cosets $\{v+H\}$ would *not* be Hausdorff. Thus, the proper categorical notion of topological vectorspace quotient, by non-closed subspace, would produce the collection of cosets $v + \overline{H}$ for the *closure* $\overline{H}$ of H.

[3.4] Claim: For a closed subspace W of a topological vectorspace V, the collection $Q = \{v + W : v \in V\}$ of cosets by W with map $q(v) = v + W$ is a topological vectorspace and q is a quotient map.

Proof: The *algebraic* quotient $Q = V/W$ of cosets $v+W$ and $q(v) = v+W$ constructs a vectorspace quotient without any topological hypotheses on W. Since W is closed, and since vector addition is a homeomorphism, $v+W$ is closed as well. Thus, its complement $V - (v+W)$ is open, so $q(V - (v+W))$ is open, by definition of the quotient topology. Thus, the complement

$$q(v) \;=\; v+W \;=\; q(v+W) \;=\; V/W - q(V - (v+W))$$

of the open set $q(V - (v+W))$ is closed. ///

Unlike general topological quotient maps,

[3.5] Claim: For a closed subspace H of a topological vector space V, the quotient map $q : V \to V/H$ is *open*, that is, carries open sets to open sets.

Proof: For U open in V,

$$q^{-1}(q(U)) \;=\; q^{-1}(U+H) \;=\; U+H \;=\; \bigcup_{h\in H} h+U$$

This is a union of opens. ///

[3.6] **Corollary:** For $f : V \to X$ a linear map with a closed subspace W of V contained in $\ker f$, and $\bar{f}$ the induced map $\bar{f} : V/W \to X$ defined by $\bar{f}(v+W) = f(v)$, f is continuous if and only if $\bar{f}$ is continuous.

Proof: Certainly if $\bar{f}$ is continuous then $f = \bar{f}\circ q$ is continuous. The converse follows from the fact that q is *open*. ///

This proves that the *construction* of the quotient by cosets succeeds in producing a quotient: a continuous linear map $f : V \to X$ *factors through* any quotient V/W for W a closed subspace contained in the kernel of f.

The notions of *balanced subset, absorbing subset*, *directed set, Cauchy net*, and *completeness* are necessary:

A subset E of V is *balanced* when $xE \subset E$ for every $x \in k$ with $|x| \le 1$.

[3.7] **Lemma:** Every neighborhood u of 0 in a topological vectorspace V over k contains a *balanced* neighborhood N of 0.

Proof: By continuity of scalar multiplication, there is $\varepsilon > 0$ and a neighborhood U' of $0 \in V$ so that if $|x| < \varepsilon$ and $v \in U'$ then $xv \in U$. Since $\mathbb{C}$ is not discrete, there is $x_o \in k$ with $0 < |x_o| < \varepsilon$. Since scalar multiplication by a non-zero element is a homeomorphism, x_oU' is a neighborhood of 0 and $x_oU' \subset U$. Put

$$N \;=\; \bigcup_{|y|\le 1} yx_oU'$$

For $|x| \le 1$, $|xy| \le |y| \le 1$, so

$$xN \;=\; \bigcup_{|y|\le 1} x(yx_oU') \subset \bigcup_{|y|\le 1} yx_oU' \;=\; N$$

producing the desired N. ///

A subset E of vectorspace V over k is *absorbing* when for every $v \in V$ there is $t_o \in R$ so that $v \in \alpha E$ for every $\alpha \in k$ so that $|\alpha| \ge t_o$.

[3.8] **Lemma:** Every neighborhood U of 0 in a topological vectorspace is *absorbing.*

Proof: We may *shrink* U to assume U is *balanced*. By continuity of the map $k \to V$ given by $\alpha \to \alpha v$, there is $\varepsilon > 0$ so that $|\alpha| < \varepsilon$ implies $\alpha v \in U$. By the *non-discreteness* of k, there is non-zero $\alpha \in k$ satisfying any such inequality. Then $v \in \alpha^{-1}U$, as desired. ///

A *poset* $S, \le$ is a partially ordered set. A *directed set* is a poset S such that, for any two elements $s, t \in S$, there is $z \in S$ so that $z \ge s$ and $z \ge t$.

A *net* in V is a subset $\{x_s : s \in S\}$ of V indexed by a directed set S. A net $\{x_s : s \in S\}$ in a topological vectorspace V is a *Cauchy net* if, for every neighborhood U of 0 in V, there is an index s_o so that for $s, t \ge s_o$ we have $x_s - x_t \in U$. A net $\{x_s : s \in S\}$ is *convergent* if there is $x \in V$ so that, for every neighborhood U of 0 in V there is an index s_o so that for $s \ge s_o$ we have $x - x_s \in U$. Since points are closed, there can be *at most* one point to which a net converges. Thus, *a convergent net is Cauchy.* Oppositely, a topological vectorspace is *complete* if every Cauchy net is convergent.

[3.9] **Lemma:** Let Y be a vector subspace of a topological vector space X, *complete* when given the subspace topology from X. Then Y is a *closed* subset of X.

Proof: Let $x \in X$ be in the closure of Y. Let S be a local basis of opens at 0, where we take the partial ordering so that $U \geq U'$ if and only if $U \subset U'$. For each $U \in S$ choose $y_U \in (x+U) \cap Y$. The net $\{y_U : U \in S\}$ converges to x, so is Cauchy. It must converge to a point in Y, so by uniqueness of limits of nets it must be that $x \in Y$. Thus, Y is closed. ///

Unfortunately, *completeness* as above is too strong a condition for general topological vectorspaces, beyond Fréchet spaces. A slightly weaker version of completeness, *quasi-completeness* or *local* completeness, *does* hold for most important natural spaces, as discussed below.

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4. *Unique vectorspace topology on* $\mathbb{C}^n$

Finite-dimensional topological vectorspaces, and their interactions with other topological vectorspaces, are especially simple:

[4.1] **Theorem:** A *finite-dimensional* complex vectorspace V has just one topological vectorspace topology, that of the product topology on $\mathbb{C}^n$ for $n = \dim V$. A finite-dimensional subspace V of a topological vectorspace W is *closed*. A $\mathbb{C}$-linear map $X \to V$ to a finite-dimensional space V is continuous if and only if the kernel is closed.

Proof: The argument is by induction. First treat the one-dimensional situation:

[4.2] **Claim:** For a one-dimensional topological vectorspace V with basis e the map $\mathbb{C} \to V$ by $x \to xe$ is a *homeomorphism*.

Proof: Since scalar multiplication is continuous, we need only show that the map is *open*. We need only do this at 0, since translation addresses other points. Given $\varepsilon > 0$, by the non-discreteness of $\mathbb{C}$ there is x_o in $\mathbb{C}$ so that $0 < |x_o| < \varepsilon$. Since V is Hausdorff, there is a neighborhood U of 0 so that $x_o e \notin U$. Shrink U so it is *balanced*. Take $x \in k$ so that $xe \in U$. For $|x| \geq |x_o|$, $|x_o x^{-1}| \leq 1$, so

$$x_o e \;=\; (x_o x^{-1})(xe) \in U$$

by balanced-ness of U, contradiction. Thus, $xe \in U$ implies that $|x| < |x_o| < \varepsilon$. ///

[4.3] **Corollary:** For fixed $x_o \in \mathbb{C}$, a not-identically-zero $\mathbb{C}$-linear $\mathbb{C}$-valued function f on V is *continuous* if and only if the *affine hyperplane* $H = \{v \in V : f(v) = x_o\}$ is *closed* in V.

Proof: Certainly if f is continuous then H is closed. For the converse, consider only the case $x_o = 0$, since translations (vector additions) are homeomorphisms of V to itself.

For v_o with $f(v_o) \neq 0$ and for any other $v \in V$

$$f(v - f(v)f(v_o)^{-1}v_o) \;=\; f(v) - f(v)f(v_o)^{-1}f(v_o) \;=\; 0$$

Thus, V/H is one-dimensional. The induced $\mathbb{C}$-linear map $\bar{f} : V/H \to k$ so that $f = \bar{f} \circ q$, that is, $\bar{f}(v+H) = f(v)$, is a homeomorphism to $\mathbb{C}$, by the previous result, so f is continuous. ///

For the theorem, for uniqueness of the topology it suffices to prove that for any $\mathbb{C}$-basis $e_1, \ldots, e_n$ for V, the map $\mathbb{C} \times \ldots \times \mathbb{C} \longrightarrow V$ by

$$(x_1, \ldots, x_n) \longrightarrow x_1 e_1 + \ldots + x_n e_n$$

is a homeomorphism. Prove this by induction on the dimension n, that is, on the number of generators for V as a free $\mathbb{C}$-module. The case $n = 1$ was treated. Since $\mathbb{C}$ is complete, the lemma above asserting the closed-ness of complete subspaces shows that any one-dimensional subspace is closed. For $n > 1$, let $H = \mathbb{C}e_1 + \ldots + \mathbb{C}e_{n-1}$. By induction, H is closed in V, so the quotient $q : V \to V/H$ is constructed as expected, as the set of cosets $v + H$. The space V/H is a one-dimensional topological vectorspace over $\mathbb{C}$, with basis $q(e_n)$. By induction, $\phi : xq(e_n) = q(xe_n) \longrightarrow x$ is a homeomorphism $V/H \to \mathbb{C}$.

Likewise, $\mathbb{C}e_n$ is a closed subspace and we have the quotient map

$$q' : V \longrightarrow V/\mathbb{C}e_n$$

The image has basis $q'(e_1), \ldots, q'(e_{n-1})$, and by induction

$$\phi' : x_1 q'(e_1) + \ldots + x_{n-1} q'(e_{n-1}) \to (x_1, \ldots, x_{n-1})$$

is a homeomorphism. By the induction hypothesis,

$$v \longrightarrow (\phi \circ q)(v) \times (\phi' \circ q')(v)$$

is continuous to $\mathbb{C}^{n-1} \times \mathbb{C} \approx \mathbb{C}^n$. On the other hand, by the continuity of scalar multiplication and vector addition,

$$\mathbb{C}^n \longrightarrow V \quad \text{by} \quad x_1 \times \ldots \times x_n \longrightarrow x_1 e_1 + \ldots + x_n e_n$$

is continuous. These two maps are mutual inverses, certifying the homeomorphism.

Thus, a n-dimensional subspace is homeomorphic to $\mathbb{C}^n$ with its *product* topology, so is complete, since a finite product of complete spaces is complete. By the closed-ness of complete subspaces, it is closed.

Continuity of a linear map $f : X \to \mathbb{C}^n$ implies that the kernel $N = \ker f$ is closed. On the other hand, for N closed, the set of cosets $x + N$ constructs a quotient, and is a topological vectorspace of dimension at most n. Therefore, the induced map $\bar{f} : X/N \to V$ is unavoidably continuous. Then $f = \bar{f} \circ q$ is continuous, where q is the quotient map. This completes the induction step. ///

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5. *Quasi-completeness*

Toward topologies in which $C_c^o(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ could be *complete*, we consider first

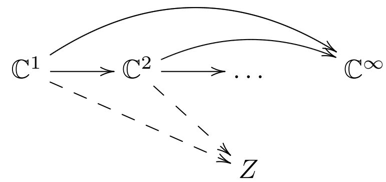
$$\mathbb{C}^\infty = \bigcup_n \mathbb{C}^n$$

where $i_n : \mathbb{C}^n \subset C^{n+1}$ by $i_n : (x_1, \ldots, x_n) \to (x_1, \ldots, x_n, 0)$. We want to topologize $\mathbb{C}^\infty$ so that it is *complete*, in a suitable sense. Above, we saw that finite-dimensional complex vectorspaces have *unique* vectorspace topologies, so the only question is how to fit them together.

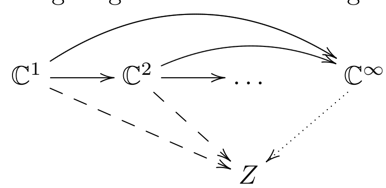
A countable ascending union of complete metric topological vector spaces, each a proper *closed* subspace of the next, such as $C^\infty = \bigcup \mathbb{C}^n$, *cannot* be a complete *metric* space, because it is exactly *presented* as a countable union of nowhere-dense closed subsets, contradicting the conclusion of the Baire Category Theorem. The function spaces $C_c^o(\mathbb{R})$ and $C_c^\infty(\mathbb{R})$ are also of this type, being the ascending unions of spaces C_K^o or C_K^∞, continuous or smooth functions with supports inside compact $K \subset \mathbb{R}$.

Thus, we cannot hope to give such space *metric* topologies for which they are *complete*.

Nevertheless, ascending unions are a type of *colimit*, just as descending intersections are a type of *limit*. That is, the topology on $\mathbb{C}^\infty$ is characterized by a universal property: for every collection of maps $f_n : \mathbb{C}^n \to Z$ with the compatibility $i_n \circ f_n = f_{n+1}$, there is a unique $f : \mathbb{C}^\infty \to Z$ through which all f_n's *factor*. That is, given a commutative diagram



there is a *unique* (dotted) map $\mathbb{C}^\infty \to Z$ giving a commutative diagram



To argue that an ascending union $X = \bigcup_n X_n$ with $X_1 \subset X_2 \subset \ldots$ is an example of a colimit, observe that every $x \in X$ lies in some X_n, so all values $f(x)$ for a map $f : X \to Z$ are completely determined by the restrictions of f to the limitands X_n. Thus, on one hand, given a compatible family $f_n : X_n \to Z$, there is *at most one* compatible $f : X \to Z$. On the other hand, a compatible family $f_n : X_n \to Z$ *defines* a map $X \to Z$: given $x \in X$, take n sufficiently large so that $x \in X_n$, and define $f(x) = f_n(x)$. The compatibility assures that it doesn't matter which sufficiently large n we use.

For the topology of $\mathbb{C}^\infty$, the colimit characterization has a possibly-counterintuitive consequence:

[5.1] Claim: *Every linear map from the space $\mathbb{C}^\infty = \mathrm{colim}_n \mathbb{C}^n$ with the colimit topology to any topological vectorspace is continuous.*

Proof: Given arbitrary linear $f : \mathbb{C}^\infty \to Z$, composition with inclusion gives a compatible family of linear maps $f_n : \mathbb{C}^n \to Z$. From above, *every* linear map from a finite-dimensional space is *continuous*. The collection $\{f_n\}$ induces a unique *continuous* map $F : \mathbb{C}^\infty \to Z$ such that $F \circ i_n : \mathbb{C}^n \to Z$ is the same as $f \circ i_n$. In general, this might not be force $f = F$. However, because X is an ascending union, the values of both F and f are completely determined by their values on the limitands, and these are the same. Thus, $f = F$. ///

The *uniqueness* argument for locally convex colimits of locally convex topological vectorspaces, that there is *at most one* such topology, is identical to the uniqueness argument for *limits*, with arrows reversed.

[5.2] Remark: The fact that a colimit of finite-dimensional spaces has a unique canonical topology, from which *every* linear map from such a colimit is *continuous*, is often misunderstood and misrepresented as suggesting that there is *no* topology on that colimit. Again, there is a *unique canonical* topology, from which every *linear* map is *continuous*.

To prove *existence* of colimits, just as limits are *subobjects* of products, colimits are *quotients* of *coproducts*, as follows. A *locally convex colimit* of topological vector spaces X_α with *transition maps* $j^\alpha_\beta : X_\alpha \to X_\beta$ is the *quotient* of the *locally convex coproduct* X of the X_α by the *closure* of the subspace Z spanned by vectors

$$j_\alpha(x_\alpha) - (j_\beta \circ j^\alpha_\beta)(x_\alpha) \qquad \text{(for all } \alpha < \beta \text{ and } x_\alpha \in X_\alpha)$$

Annihilation of these differences in the quotient forces the desired compatibility relations. Obviously, quotients of locally convex spaces are locally convex.

Locally convex coproducts X of topological vector spaces X_α are coproducts (also called *direct sums*) of the *vector spaces* X_α topologized by the *diamond topology*, described as follows. [2] For a collection U_α of convex neighborhoods of 0 in the X_α, let

$$U \;=\; \text{convex hull in } X \text{ of the union of } j_\alpha(U_\alpha) \qquad \text{(with } j_\alpha : X_\alpha \to X \text{ the } \alpha^{th} \text{ canonical map)}$$

[2] The *product* topology of locally convex topological vector spaces is locally convex, whether in the category of locally convex topological vector spaces or in the larger category of not-necessarily-locally-convex topological vector

The diamond topology has local basis at 0 consisting of such U. Thus, it is locally convex by construction. *Closedness of points* follows from the corresponding property of the X_α. Thus, *existence* of a *locally convex* coproduct of locally convex spaces is assured by the *construction*.

A *countable* colimit of a family $V_1 \to V_2 \to \ldots$ of topological vectorspaces is a *strict* colimit, or *strict inductive limit*, when each $V_i \to V_{i+1}$ is an isomorphism to its image, and each image is closed. A strict colimit of *Fréchet* spaces is called an *LF-space*.

Just to be sure:

[5.3] Claim: In a colimit indexed by positive integers $V = \mathrm{colim} V_i$, if every transition $V_i \to V_{i+1}$ is *injective*, then every limitand V_i *injects* to the colimit V. Further, the colimit is the ascending union of the limitands V_i, suitably topologized.

Proof: In effect, the argument presents the colimit corresponding to an ascending union more directly, not as a quotient of the coproduct, although it is convenient to already have *existence* of the colimit. Certainly each V_i injects to $W = \bigcup_n V_n$. We will give W a locally convex topology so that every inclusion $V_i \to W$ is continuous. The universal property of the colimit produces a unique compatible map $V \to W$, so every V_i must inject to V itself.

Since the maps j_i of V_i to the colimit V are injections, the ascending union W injects to V by $j(w) = j_i(w)$ for any index i large enough so that $w \in V_i$. The compatibility of the maps among the V_i assures that j is well-defined. We claim that $j(W)$ with the subspace topology from V, and the inclusions $V_i \to j_i(V_i) \subset j(W)$, give a colimit of the V_i. Indeed for any compatible, family $f_i : V_i \to Z$ and induced $f : V \to Z$, the restriction of f to $j(W)$ gives a map $j(W) \to Z$ through which the f_i factor. Thus, in fact, such a colimit is the ascending union with a suitable topology.

Now we describe a topology on the ascending union W so that all inclusions $V_i \to W$ are continuous. Give W a local basis $\{U\}$ at 0, by taking arbitrary convex opens $U_i \subset V_i$ containing 0, and letting U be the convex hull of $\bigcup_i U_i$. Every injection $V_i \to W$ is continuous, because the inverse image of such $U \cap V_i$ contains U_i, giving continuity at 0.

To be sure that *points are closed* in W, given $0 \neq x \in W$, we find a neighborhood of 0 in W not containing x. Let i_o be the first index such that $x \in V_{i_o}$. By Hahn-Banach, there is a continuous linear functional λ_{i_o} on V_{i_o} such that $\lambda_{i_o}(x) \neq 0$. Without loss of generality, $\lambda_{i_o}(x) = 1$ and $|\lambda_{i_o}| = 1$. Use Hahn-Banach to extend λ_{i_o} to a continuous linear functional λ_i on V_i for every $i \geq i_o$, with $|\lambda_i| \leq 1$. λ_{i_o} gives a continuous linear functional on V_i for $i < i_o$ by composition with the injection $V_i \to V_{i_o}$. Then $U_i = \{y \in V_i : |\lambda_i(y)| < 1\}$ is open in V_i and does not contain x, for all i. The convex hull of the ascending union $\bigcup_i U_i$ is just $\bigcup_i U_i$ itself, so does not contain x.

We did not quite prove that this topology is exactly the colimit topology, but we will never need that fact. ///

Typical *colimit* topologies are *not* complete in the strongest possible sense (see below), but are *quasi-complete*, a property sufficient for all applications. To describe quasi-completeness, we need a notion of *boundedness* in general topological vectorspaces, not merely metrizable ones. A subset B of a topological vector space V is *bounded* when, for every open neighborhood N of 0 there is $t_o > 0$ such that $B \subset tN$ for every $t \geq t_o$. A space is *quasi-complete* when every *bounded* Cauchy *net* is *convergent*.

Nothing new for metric spaces:

[5.4] Lemma: Complete metric spaces are quasi-complete. In particular, Cauchy nets converge, and contain

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spaces. However, *coproducts* behave differently: the locally convex coproduct of *uncountably many* locally convex spaces is *not* a coproduct in the larger category of not-necessarily-locally-convex spaces. This already occurs with an uncountable coproduct of *lines*.

cofinal *sequences* converging to the same limit.

Proof: Let $\{s_i : i \in I\}$ be a Cauchy net in X. Given a natural number n, let $i_n \in I$ be an index such that $d(x_i, x_j) < \frac{1}{n}$ for $i, j \geq i_n$. Then $\{x_{i_n} : n = 1, 2, \ldots\}$ is a Cauchy sequence, with limit x. Given $\varepsilon > 0$, let $j \geq i_n$ be also large enough such that $d(x, x_j) < \varepsilon$. Then

$$d(x, x_{i_n}) \leq d(x, x_j) + d(x_j, x_{i_n}) < \varepsilon + \frac{1}{n} \qquad \text{(for every } \varepsilon > 0)$$

Thus, $d(x, x_{i_n}) \leq \frac{1}{n}$. The original Cauchy net also converges to x: given $\varepsilon > 0$, for n large enough so that $\varepsilon > \frac{1}{n}$,

$$d(x_i, x) \ \leq \ d(x_i, x_{i_n}) + d(x_{i_n}, x) \ < \ \varepsilon + \varepsilon \qquad \text{(for } i \geq i_n)$$

with the strict inequality coming from $d(x_{i_n}, x) < \varepsilon$. ///

[5.5] Theorem: A bounded subset of an LF-space $X = \mathrm{colim}_n X_n$ lies in some limitand X_n. An LF-space is *quasi-complete*.

Proof: Let B be a bounded subset of X. Suppose B does *not* lie in any X_i. Then there is a sequence $i_1, i_2, \ldots$ of positive integers and x_{i_ℓ} in $X_{i_\ell} \cap B$ with x_{i_ℓ} *not* lying in $X_{i_\ell - 1}$. Using $X = \bigcup_j X_{i_\ell}$, without loss of generality, suppose that $i_\ell = \ell$.

By the Hahn-Banach theorem and induction, using the closedness of X_{i-1} in X_i, there are continuous linear functionals λ_i on X_i's such that $\lambda_i(x_i) = i$ and the restriction of λ_i to X_{i-1} is λ_{i-1}, for example. Since X is the colimit of the X_i, this collection of functionals exactly describes a unique compatible continuous linear functional λ on X.

But $\lambda(B)$ is *bounded* since B is bounded and λ is continuous, precluding the possibility that λ takes on all positive integer values at the points x_i of B. Thus, it could *not* have been that B failed to lie inside some single X_i. The strictness of the colimit implies that B is bounded as a subset of X_i, proving one direction of the equivalence. The other direction of the equivalence is less interesting.

Thus a *bounded* Cauchy net lies in some limitand Fréchet space X_n, so is convergent there, since Fréchet spaces are complete. ///

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6. *Seminorms and locally convex topologies*

The simplest vectorspace topologies are Banach (or Hilbert) spaces, limits of Banach spaces, and colimits of limits of Banach spaces. By design, these descriptions facilitate proof of (quasi-) *completeness*. *Weaker* topologies are not usually described in this fashion. For example, for a topological vectorspace V, with (continuous) *dual*

$$V^* = \{\text{continuous linear maps } V \to \mathbb{C}\}$$

the *weak dual topology*[3] on V^* has a local sub-basis at 0 consisting of sets

$$U \ = \ U_{v,\varepsilon} \ = \ \{\lambda \in V^* : |\lambda(v)| < \varepsilon\} \qquad \text{(for fixed } v \in V \text{ and } \varepsilon > 0)$$

Unless V is finite-dimensional, this topology on V^* is much coarser than a Banach, Fréchet, or LF-topology. The map $\lambda \to |\lambda(v)|$ is a natural example of a *seminorm*. It is not a norm, because $\lambda(v) = 0$ can easily happen.

Seminorms are a general device to describe topologies on vectorspaces. These topologies are invariably *locally convex*, in the sense of having a local basis at 0 consisting of *convex* sets.

[3] The weak dual topology is traditionally called the *weak-*-topology*, but replacing * by *dual* is more explanatory.

Description of a vectorspace topology by seminorms does *not* generally give direct information about *completeness*. Nevertheless, we can prove *quasi-completeness* for an important class of examples, just below.

A *seminorm* ν on a complex vectorspace V is a real-valued function on V so that $\nu(x) \geq 0$ for all $x \in V$ *(non-negativity)*, $\nu(\alpha x) = |\alpha| \cdot \nu(x)$ for all $\alpha \in \mathbb{C}$ and $x \in V$ *(homogeneity)*, and $\nu(x+y) \leq \nu(x) + \nu(y)$ for all $x, y \in V$ *(triangle inequality)*. This differs from the notion of *norm* only in the significant point that we allow $\nu(x) = 0$ for $x \neq 0$.

To compensate for the possibility that an individual seminorm can be 0 on a particular non-zero vector, since we want Hausdorff topologies, we mostly care about *separating* families $\{\nu_i : i \in I\}$ of semi-norms: for every $0 \neq x \in V$ there is ν_i so that $\nu_i(x) \neq 0$.

[6.1] Claim: The collection Φ of all *finite intersections* of sets

$$U_{i,\varepsilon} \;=\; \{x \in V : \nu_i(x) < \varepsilon\} \qquad\qquad (\text{for } \varepsilon > 0 \text{ and } i \in I)$$

is a *local basis* at 0 for a locally convex topology on V.

Proof: As expected, we intend to define a topological vector space topology on V by saying a set U is *open* if and only if for every $x \in U$ there is some $N \in \Phi$ so that $x + N \subset U$ This would be the *induced topology* associated to the family of seminorms.

That we have a *topology* does not use the hypothesis that the family of seminorms is *separating*, although points will not be closed without the separating property. Arbitrary unions of sets containing sets of the form $x + N$ containing each point x have the same property. The empty set and the whole space V are visibly in the collection. The least trivial issue is to check that finite intersections of such sets are again of the same form. Looking at each point x in a given finite intersection, this amounts to checking that finite intersections of sets in Φ are again in Φ. But Φ is *defined* to be the collection of all finite intersections of sets $U_{i,\varepsilon}$, so this succeeds: we have closure under finite intersections, and a topology on V.

To verify that this topology makes V a topological vectorspace is to verify the continuity of vector addition and continuity of scalar multiplication, and closed-ness of points. None of these verifications is difficult:

The *separating* property implies that for each $x \in V$ the intersection of *all* the sets $x + N$ with $N \in \Phi$ is just x. Given $y \in V$, for each $x \neq y$ let U_x be an open set containing x but not y. Then

$$U \;=\; \bigcup_{x \neq y} U_x$$

is *open* and has complement $\{y\}$, so the singleton $\{y\}$ is *closed*.

For continuity of vector addition, it suffices to prove that, given $N \in \Phi$ and given $x, y \in V$ there are $U, U' \in \Phi$ so that

$$(x + U) + (y + U') \;\subset\; x + y + N$$

The triangle inequality implies that for a fixed index i and for $\varepsilon_1, \varepsilon_2 > 0$

$$U_{i,\varepsilon_1} + U_{i,\varepsilon_2} \;\subset\; U_{i,\varepsilon_1+\varepsilon_2}$$

Then

$$(x + U_{i,\varepsilon_1}) + (y + U_{i,\varepsilon_2}) \;\subset\; (x+y) + U_{i,\varepsilon_1+\varepsilon_2}$$

Thus, given

$$N \;=\; U_{i_1,\varepsilon_1} \cap \ldots \cap U_{i_n,\varepsilon_n}$$

take

$$U \;=\; U' \;=\; U_{i_1,\varepsilon_1/2} \cap \ldots \cap U_{i_n,\varepsilon_n/2}$$

proving continuity of vector addition.

For continuity of scalar multiplication, prove that for given $\alpha \in k$, $x \in V$, and $N \in \Phi$ there are $\delta > 0$ and $U \in \Phi$ so that

$$(\alpha + B_\delta) \cdot (x + U) \ \subset\ \alpha x + N \qquad\qquad (\text{with } B_\delta = \{\beta \in k : |\alpha - \beta| < \delta\})$$

Since N is an intersection of the sub-basis sets $U_{i,\varepsilon}$, it suffices to consider the case that N is such a set. Given α and x, for $|\alpha' - \alpha| < \delta$ and for $x - x' \in U_{i,\delta}$,

$$\begin{aligned}\nu_i(\alpha x - \alpha' x') \ &=\ \nu_i((\alpha - \alpha')x + (\alpha'(x - x')) \ \le\ \nu_i((\alpha - \alpha')x) + \nu_i(\alpha'(x - x'))\\ &=\ |\alpha - \alpha'| \cdot \nu_i(x) + |\alpha'| \cdot \nu_i(x - x') \ \le\ |\alpha - \alpha'| \cdot \nu_i(x) + (|\alpha| + \delta) \cdot \nu_i(x - x')\\ &\le\ \delta \cdot (\nu_i(x) + |\alpha| + \delta)\end{aligned}$$

Thus, for the joint continuity, take $\delta > 0$ small enough so that

$$\delta \cdot (\delta + \nu_i(x) + |\alpha|) \ <\ \varepsilon$$

Taking finite intersections presents no further difficulty, taking the corresponding finite intersections of the sets B_δ and $U_{i,\delta}$, finishing the demonstration that separating families of seminorms give a structure of topological vectorspace.

Last, check that finite intersections of the sets $U_{i,\varepsilon}$ are *convex*. Since intersections of convex sets are convex, it suffices to check that the sets $U_{i,\varepsilon}$ themselves are convex, which follows from the homogeneity and the triangle inequality: with $0 \le t \le 1$ and $x, y \in U_{i,\varepsilon}$,

$$\nu_i(tx + (1-t)y) \ \le\ \nu_i(tx) + \nu_i((1-t)y) \ =\ t\nu_i(x) + (1-t)\nu_i(y) \ \le\ t\varepsilon + (1-t)\varepsilon \ =\ \varepsilon$$

Thus, the set $U_{i,\varepsilon}$ is convex. ///

The converse, that *every* locally convex topology is given by a family of seminorms, is more difficult:

Let U be a *convex* open set containing 0 in a topological vectorspace V. Every open neighborhood of 0 contains a *balanced* neighborhood of 0, so shrink U if necessary so it is balanced, that is, $\alpha v \in U$ for $v \in U$ and $|\alpha| \le 1$. The *Minkowski functional* ν_U associated to U is

$$\nu_U(v) \ =\ \inf\{t \ge 0 : v \in tU\}$$

[6.2] Claim: The Minkowski functional ν_U associated to a balanced convex open neighborhood U of 0 in a topological vectorspace V is a *seminorm* on V, and is *continuous* in the topology on V.

Proof: The argument is as expected:

By continuity of scalar multiplication, *every* neighborhood U of 0 is *absorbing*, in the sense that every $v \in V$ lies inside tU for large enough $|t|$. Thus, the set over which we take the infimum to define the Minkowski functional is *non-empty*, so the infimum exists.

Let α be a scalar, and let $\alpha = s\mu$ with $s = |\alpha|$ and $|\mu| = 1$. The balanced-ness of U implies the balanced-ness of tU for any $t \ge 0$, so for $v \in tU$ also

$$\alpha v \ \in\ \alpha t U \ =\ s\mu t U \ =\ stU$$

From this,

$$\{t \ge 0 : \alpha v \in \alpha U\} \ =\ |\alpha| \cdot \{t \ge 0 : \alpha v \in tU\}$$

from which follows the *homogeneity* property required of a seminorm:

$$\nu_U(\alpha v) \;=\; |\alpha| \cdot \nu_U(v) \qquad\qquad \text{(for scalar } \alpha\text{)}$$

For the triangle inequality use the convexity. For $v, w \in V$ and $s, t > 0$ such that $v \in sU$ and $w \in tU$,

$$v + w \;\in\; sU + tU \;=\; \{su + tu' \;:\; u, u' \in U\}$$

By convexity,

$$su + tu' \;=\; (s+t) \cdot \Big(\frac{s}{s+t} \cdot u + \frac{t}{s+t} \cdot u'\Big) \;\in\; (s+t) \cdot U$$

Thus,

$$\nu_U(v+w) \;=\; \inf\{r \geq 0 : v+w \in rU\} \;\leq\; \inf\{r \geq 0 : v \in rU\} + \inf\{r \geq 0 : w \in rU\} \;=\; \nu_U(v) + \nu_U(w)$$

Thus, the Minkowski functional ν_U attached to balanced, convex U is a continuous seminorm. ///

[6.3] Theorem: The topology of a *locally convex* topological vectorspace V is given by the collection of seminorms obtained as Minkowski functionals ν_U associated to a local basis at 0 consisting of convex, balanced opens.

Proof: The proof is straightforward, once we decide to tolerate an extravagantly large collection of seminorms. With or without local convexity, every neighborhood of 0 contains a *balanced* neighborhood of 0. Thus, a locally convex topological vectorspace has a local basis X at 0 of *balanced convex* open sets.

We claim that every open $U \in X$ can be recovered from the corresponding seminorm ν_U by

$$U \;=\; \{v \in V \;:\; \nu_U(v) < 1\}$$

Indeed, for $v \in U$, the continuity of scalar multiplication gives $\delta > 0$ and a neighborhood N of v such that $z \cdot v - 1 \cdot v \in U$ for $|1 - z| < \delta$. Thus, $v \in (1+\delta)^{-1} \cdot U$, so

$$\nu_U(v) \;=\; \inf\{t \geq 0 \;:\; v \in t \cdot U\} \;\leq\; \frac{1}{1+\delta} \;<\; 1$$

On the other hand, for $\nu_U(v) < 1$, there is $t < 1$ such that $v \in tU \subset U$, since U is convex and contains 0. Thus, the seminorm topology is at least as fine as the original.

Oppositely, the same argument shows that every seminorm local basis open

$$\{v \in V \;:\; \nu_U(v) < t\}$$

is simply tU. Thus, the original topology is at least as fine as the seminorm topology. ///

The comparison of descriptions of topologies is straightforward, as follows. For a seminorm ν on a topological vectorspace V, we can form a Banach space *completing* with respect to the *pseudo-metric* $\nu(x - y)$. In particular, unlike completions with respect to genuine metrics, there can be *collapsing*, so that the natural map of V to this completion need not be an injection.

[6.4] Claim: Let V be a topological vectorspace with topology given by a (separating) family of seminorms $S = \{\nu\}$. Order the set of finite subsets of S by inclusion, and

$$\nu_F \;=\; \sum_{\nu \in F} \nu \qquad\qquad \text{(for finite subset } F \text{ of } S\text{)}$$

Then V with its seminorm topology is a dense subspace of the limit $\lim_{F\in\Phi} V_F$ of the Banach-space completions V_F with respect to ν_F.

Proof: As earlier, the seminorm topology is literally the subspace topology on the diagonal copy of V in the product of the V_F.

Of course, the poset of finite subsets of S is more complicated than the poset of positive integers, so such a limit can be large. Certainly V has a natural map to every V_F. Indeed, by definition of the seminorm topology, the open sets in V are exactly the inverse images in V of open sets in the various V_F.

For $F \subset F'$, since $\nu_{F'} \geq \nu_F$, there is a natural continuous linear map $V_{F'} \to V_F$. The maps $V \to V_F$ are compatible, in the sense that the composite $V \to V_{F'} \to V_F$ is the same as $V \to V_F$, for $F \subset F'$. This induces a unique continuous linear map of V to the limit of the V_F.

The limit is the diagonal

$$D \;=\; \{\{v_F\} \in \prod_F V_F : v_{F'} \to v_F, \text{ for all } F' \supset F\} \;\;\subset\;\; \prod_F V_F$$

with the subspace topology. Repeating part of an earlier argument, given a finite collection of finite subsets $F_1, \ldots, F_n$ of S, for $\{v_F\} \in D$, take neighborhoods $U_i \subset V_{F_i}$ containing v_{F_i}. Let $\Phi = \bigcup_i F_i$. The compatibility implies that there is $v_\Phi \in V_\Phi$ such that $v_\Phi \to v_{F_i}$ for all i. Also, there is a sufficiently small neighborhood U of v_Φ such that its image in every V_{F_i} is inside the neighborhood U_i of v_{F_i}. Since the image of V is dense in V_Φ, take $v \in V$ with image inside U. Then the image of v is inside U_i for all i. Thus, the image of V is dense in the limit. ///

Although it turns out that we only care about locally convex topological vectorspaces, there *do exist* complete-metric topological vectorspaces which *fail* to be locally convex. This underscores the need to explicitly specify that a Fréchet space should be locally convex. The usual example of a not-locally-convex complete-metric space is the sequence space

$$\ell^p \;=\; \{x = (x_1, x_2, \ldots) \;:\; \sum_i |x_i|^p < \infty\}$$

for $0 < p < 1$ with metric

$$d(x,y) \;=\; \sum_i |x_i - y_i|^p \qquad\qquad \text{(note: no } p^{th} \text{ root, unlike the } p \geq 1 \text{ case)}$$

This example's interest is mostly as a counterexample to a naive presumption that local convexity is automatic.

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7. *Quasi-completeness theorem*

We have already seen that LF-spaces such as the space of test functions $\mathcal{D}(\mathbb{R}) = C_c^\infty(\mathbb{R})$, although *not* complete metrizable, are *quasi-complete*. It is fortunate that most important topological vector spaces are quasi-complete.

At the end of this section, we show that the fullest notion of completeness easily fails to hold, even for quasi-complete spaces.

It is clear that *closed subspaces* of quasi-complete spaces are quasi-complete. Products and finite sums of quasi-complete spaces are quasi-complete.

Let $\mathrm{Hom}(X,Y)$ be the space of continuous linear functions from a topological vectorspace X to another topological vectorspace Y. Give $\mathrm{Hom}(X,Y)$ the topology by seminorms $p_{x,U}$ where $x \in X$ and U is a convex, balanced neighborhood of 0 in Y, defined by

$$p_{x,U}(T) \;=\; \inf\{t > 0 : Tx \in tU\} \qquad\qquad (\text{for } T \in \mathrm{Hom}(X,Y))$$

For $Y = \mathbb{C}$, this gives the weak dual topology on X^*.

[7.1] Theorem: For X a Fréchet space or LF-space, and Y quasi-complete, the space $\mathrm{Hom}(X, Y)$, with the topology induced by the seminorms $p_{x,U}$, is *quasi-complete.*

Proof: Some preparation is required. A set E of continuous linear maps from one topological vectorspace X to another topological vectorspace Y is *equicontinuous* when, for every neighborhood U of 0 in Y, there is a neighborhood N of 0 in X so that $T(N) \subset U$ for every $T \in E$.

[7.2] Claim: Let V be a strict colimit of a well-ordered countable collection of locally convex closed subspaces V_i. Let Y be a locally convex topological vectorspace. Let E be a set of continuous linear maps from V to Y. Then E is *equicontinuous* if and only if for each index i the collection of continuous linear maps $\{T|_{V_i} : T \in E\}$ is equicontinuous.

Proof: Given a neighborhood U of 0 in Y, shrink U if necessary so that U is convex and balanced. For each index i, let N_i be a convex, balanced neighborhood of 0 in V_i so that $TN_i \subset U$ for all $T \in E$. Let N be the convex hull of the union of the N_i. By the convexity of N, still $TN \subset U$ for all $T \in E$. By the construction of the diamond topology, N is an open neighborhood of 0 in the coproduct, hence in the colimit, which is a quotient of the coproduct. This gives the equicontinuity of E. The other direction of the implication is easy. ///

[7.3] Claim: (*Banach-Steinhaus/uniform boundedness theorem*) Let X be a Fréchet space or LF-space and Y a locally convex topological vector space. A set E of linear maps $X \to Y$, such that every set $Ex = \{Tx : T \in E\}$ is *bounded* in Y, is *equicontinuous.*

Proof: First consider X Fréchet. Given a neighborhood U of 0 in Y, let $A = \bigcap_{T \in E} T^{-1}\overline{U}$. By assumption, $\bigcup_n nA = X$. By the Baire category theorem, the complete metric space X is not a countable union of nowhere dense subsets, so at least one of the closed sets nA has non-empty interior. Since (non-zero) scalar multiplication is a homeomorphism, A itself has non-empty interior, containing some $x + N$ for a neighborhood N of 0 and $x \in A$. For every $T \in E$,

$$TN \;\subset\; T\{a - x : a \in A\} \;\subset\; \{u_1 - u_2 \;:\; u_1, u_2 \in \overline{U}\} \;=\; \overline{U} - \overline{U}$$

By continuity of addition and scalar multiplication in Y, given an open neighborhood U_o of 0, there is U such that $\overline{U} - \overline{U} \subset U_o$. Thus, $TN \subset U_o$ for every $T \in E$, and E is equicontinuous.

For $X = \bigcup_i X_i$ an LF-space, this argument already shows that E restricted to each X_i is equicontinuous. As in the previous claim, this gives equicontinuity on the strict colimit. ///

For the proof of the theorem on quasi-completeness, let $E = \{T_i : i \in I\}$ be a bounded Cauchy net in $\mathrm{Hom}(X, Y)$, where I is a directed set. Of course, attempt to define the limit of the net by $Tx = \lim_i T_i x$. For $x \in X$ the evaluation map $S \to Sx$ from $\mathrm{Hom}(X, Y)$ to Y is continuous. In fact, the topology on $\mathrm{Hom}(X, Y)$ is the coarsest with this property. Therefore, by the quasi-completeness of Y, for each fixed $x \in X$ the net $T_i x$ in Y is bounded and Cauchy, so converges to an element of Y suggestively denoted Tx.

To prove *linearity* of T, fix x_1, x_2 in X, $a, b \in \mathbb{C}$ and fix a neighborhood U_o of 0 in Y. Since T is in the closure of E, for any open neighborhood N of 0 in $\mathrm{Hom}(X, Y)$, there exists

$$T_i \;\in\; E \cap (T + N)$$

In particular, for any neighborhood U of 0 in Y, take

$$N = \{S \in \mathrm{Hom}(X, Y) : S(ax_1 + bx_2) \in U,\; S(x_1) \in U,\; S(x_2) \in U\}$$

Then

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2)$$

$$= (T(ax_1 + bx_2) - aT(x_1) - bT(x_2)) - (T_i(ax_1 + bx_2) - aT_i(x_1) - bT_i(x_2))$$

since T_i is linear. The latter expression is

$$T(ax_1 + bx_2) - (ax_1 + bx_2) + a(T(x_1) - T_i(x_1) + b(T(x_2) - T_i(x_2)$$

$$\in U + aU + bU$$

By choosing U small enough so that

$$U + aU + bU \subset U_o$$

we find that

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) \in U_o$$

Since this is true for every neighborhood U_o of 0 in Y,

$$T(ax_1 + bx_2) - aT(x_1) - bT(x_2) = 0$$

which proves linearity.

Continuity of the limit operator T exactly requires *equicontinuity* of $E = \{T_i x : i \in I\}$. Indeed, for each $x \in X$, $\{T_i x : i \in I\}$ is *bounded* in Y, so by Banach-Steinhaus, $\{T_i : i \in I\}$ is equicontinuous.

Fix a neighborhood U of 0 in Y. Invoking the equicontinuity of E, let N be a small enough neighborhood of 0 in X so that $T(N) \subset U$ for all $T \in E$. Let $x \in N$. Choose an index i sufficiently large so that $Tx - T_i x \in U$, vis the definition of the topology on $\mathrm{Hom}(X, Y)$. Then

$$Tx \in U + T_i x \subset U + U$$

The usual rewriting, replacing U by U' such that $U' + U' \subset U$, shows that T is continuous. ///

Finally, we demonstrate that weak duals of reasonable topological vector spaces, such as infinite-dimensional Hilbert, Banach, or Fréchet spaces, are definitely *not* complete in the strongest sense. That is, in these weak duals there are Cauchy *nets* which *do not converge*.

[7.4] Theorem: The weak dual of a locally-convex topological vector space V is *complete* if and only if every linear functional on V is *continuous*.

Proof: A vectorspace V can be (re-) topologized as the colimit V_{init} of all its finite-dimensional subspaces. Although the poset of finite-dimensional subspaces is much larger than the poset of positive integers, the argument still applies: this colimit really is the ascending union with a suitable topology.

[7.5] Claim: For a locally-convex topological vector space V the identity map $V_{\mathrm{init}} \to V$ is *continuous*. That is, V_{init} is the finest locally convex topological vector space topology on V.

Proof: Finite-dimensional topological vector spaces have unique topologies. Thus, for any finite-dimensional vector subspace X of V the inclusion $X \to V$ is continuous with that unique topology on X. These inclusions form a compatible family of maps to V, so by the characterization of colimit there is a *unique* continuous map $V_{\mathrm{init}} \to V$. This map is the identity on every finite-dimensional subspace, so is the identity on the underlying set V. ///

[7.6] Claim: Every linear functional $\lambda : V_{\mathrm{init}} \to \mathbb{C}$ is *continuous*.

Proof: The restrictions of a given linear function λ on V to finite-dimensional subspaces are compatible with the inclusions among finite-dimensional subspaces. Every linear functional on a finite-dimensional space is continuous, so the characterizing property of the colimit implies that λ is continuous on V_{init}. ///

[7.7] Claim: The weak dual V^* of a locally-convex topological vector space V injects continuously to the limit of the finite-dimensional Banach spaces

$$V^*_\Phi \;=\; \text{completion of } V^* \text{ under seminorm } \; p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad\qquad (\text{finite } \Phi \subset V)$$

and the weak dual topology is the subspace topology.

Proof: The weak dual topology on the continuous dual V^* of a topological vector space V is given by the seminorms

$$p_v(\lambda) \;=\; |\lambda(v)| \qquad\qquad (\text{for } \lambda \in V^* \text{ and } v \in V)$$

The corresponding local basis is finite intersections

$$\{\lambda \in V^* \;:\; |\lambda(v)| < \varepsilon, \text{ for all } v \in \Phi\} \qquad\qquad (\text{for arbitrary finite sets } \Phi \subset V)$$

These sets contain, and are contained in, sets of the form

$$\{\lambda \in V^* \;:\; \sum_{v\in\Phi} |\lambda(v)| < \varepsilon\} \qquad\qquad (\text{for arbitrary finite sets } \Phi \subset V)$$

Therefore, the weak dual topology on V^* is also given by semi-norms

$$p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad\qquad (\text{finite } \Phi \subset V)$$

These have the convenient feature that they form a projective family, indexed by (reversed) inclusion. Let $V^*(\Phi)$ be V^* with the p_Φ-topology: this is not Hausdorff, so continuous linear maps $V^* \to V^*(\Phi)$ descend to maps $V^* \to V^*_\Phi$ to the *completion* V^*_Φ of V^* with respect to the pseudo-metric attached to p_Φ. The quotient map $V^*(\Phi) \to V^*_\Phi$ typically has a large kernel, since

$$\dim_{\mathbb{C}} V^*_\Phi \;=\; \text{card}\,\Phi \qquad\qquad (\text{for finite } \Phi \subset V)$$

The maps $V^* \to V^*_\Phi$ are compatible with respect to (reverse) inclusion $\Phi \supset Y$, so V^* has a natural induced map to the $\lim_\Phi V^*_\Phi$. Since V separates points in V^*, V^* *injects* to the limit. The weak topology on V^* is exactly the subspace topology from that limit. ///

[7.8] Claim: The weak dual V^*_{init} of V_{init} is the limit of the finite-dimensional Banach spaces

$$V^*_\Phi \;=\; \text{completion of } V^*_{\text{init}} \text{ under seminorm } \; p_\Phi(\lambda) \;=\; \sum_{v\in\Phi} |\lambda(v)| \qquad\qquad (\text{finite } \Phi \subset V)$$

Proof: The previous proposition shows that V^*_{init} *injects* to the limit, and that the subspace topology from the limit is the weak dual topology. On the other hand, the limit consists of linear functionals on V, without regard to topology or continuity. Since *all* linear functionals are continuous on V_{init}, the limit is naturally a subspace of V^*_{init}. ///

Returning to the proof of the theorem, $\lim_\Phi V^*_\Phi$ is a closed subspace of the corresponding *product*, so is *complete* in the fullest sense. Any other locally convex topologization V_τ of V has weak dual $(V_\tau)^* \subset (V_{\text{init}})^*$ with the subspace topology, and the image is *dense* in $(V_{\text{init}})^*$. Thus, unless $(V_\tau)^* = (V_{\text{init}})^*$, the weak dual V^*_τ *is not complete.* ///