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Hilbert-Schmidt operators, nuclear spaces, kernel theorem I

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1. Hilbert-Schmidt operators
2. Simplest nuclear Fréchet spaces
3. Strong dual topologies and colimits
4. Schwartz' kernel theorem for Sobolev spaces
5. Appendix: joint continuity of bilinear maps on Fréchet spaces

Hilbert-Schmidt operators on Hilbert spaces are especially simple compact operators.

Countable projective limits of Hilbert spaces with transition maps Hilbert-Schmidt constitute the simplest class of *nuclear spaces*: well-behaved with respect to *tensor products* and other natural constructs.

The main application in mind is proof of *Schwartz' Kernel Theorem* in the important example of L^2 Sobolev spaces.

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1. *Hilbert-Schmidt operators*

[1.1] Prototype: integral operators

For a continuous function $Q(a, b)$ on $[a, b] \times [a, b]$, define $T : L^2[a, b] \to L^2[a, b]$ by

$$Tf(y) \;=\; \int_a^b Q(x, y)\, f(x)\, dx$$

The function Q is the (integral) *kernel* of T. [1] Approximating Q by *finite* linear combinations of 0-or-1-valued functions shows that T is a uniform operator norm limit of finite-rank operators, so is *compact*. In fact, T falls into an even-nicer sub-class of compact operators, the *Hilbert-Schmidt* operators, as in the following.

[1.2] Hilbert-Schmidt norm on $V \otimes_{\text{alg}} W$

In the category of Hilbert spaces and continuous linear maps, demonstrably there is *no* tensor product in the categorical sense. [2] *Not* claiming anything about genuine tensor products in any category of topological vector spaces, the *algebraic* tensor product $X \otimes_{\text{alg}} Y$ of two Hilbert spaces has a hermitian inner product $\langle , \rangle_{\text{HS}}$ determined by

$$\langle x \otimes y, x' \otimes y' \rangle_{\text{HS}} \;=\; \langle x, x' \rangle \, \langle y, y' \rangle$$

Let $X \otimes_{\text{HS}} Y$ be the completion with respect to the corresponding norm $|v|_{\text{HS}} = \langle v, v \rangle_{\text{HS}}^{1/2}$

$$X \otimes_{\text{HS}} Y \;=\; | \cdot |_{\text{HS}}\text{-completion of } X \otimes_{\text{alg}} Y$$

This completion is a Hilbert space.

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[1] Yes, the use of *kernel* in reference to a two-argument function integrated-against is incompatible with use of *kernel* for homomorphisms of groups or modules.

[2] See [Garrett 2010] for proof of non-existence of a Hilbert-space tensor product. The point is that not every Hilbert-Schmidt operator is of trace class. *Nuclear spaces* are a family of topological vector spaces that overcome problems with tensor products. The simplest nuclear spaces are constructed from families of Hilbert spaces connected by Hilbert-Schmidt operators, as in the first part of the discussion below.

[1.3] Hilbert-Schmidt operators

For Hilbert spaces V, W the finite-rank [3] continuous linear maps $T : V \to W$ can be identified with the algebraic tensor product $V^* \otimes_{\mathrm{alg}} W$, by [4]

$$(\lambda \otimes w)(v) \;=\; \lambda(v) \cdot w$$

The space of *Hilbert-Schmidt operators* $V \to W$ is the completion of the space $V^* \otimes_{\mathrm{HS}} W$ of finite-rank operators, with respect to the *Hilbert-Schmidt norm* $|\cdot|_{\mathrm{HS}}$ on $V^* \otimes_{\mathrm{alg}} W$. For example,

$$|\lambda \otimes w + \lambda' \otimes w'|^2_{\mathrm{HS}} \;=\; \langle \lambda \otimes w + \lambda' \otimes w', \lambda \otimes w + \lambda' \otimes w' \rangle$$

$$=\; \langle \lambda \otimes w, \lambda \otimes w\rangle + \langle \lambda \otimes w, \lambda' \otimes w'\rangle + \langle \lambda' \otimes w', \lambda \otimes w\rangle + \langle \lambda' \otimes w', \lambda' \otimes w'\rangle$$

$$=\; |\lambda|^2 |w|^2 + \langle \lambda, \lambda'\rangle \langle w, w'\rangle + \langle \lambda', \lambda\rangle\langle w', w\rangle + |\lambda'|^2|w'|^2$$

When $\lambda \perp \lambda'$ or $w \perp w'$, the monomials $\lambda \otimes w$ and $\lambda' \otimes w'$ are orthogonal, and

$$|\lambda \otimes w + \lambda' \otimes w'|^2_{\mathrm{HS}} \;=\; |\lambda|^2|w|^2 + |\lambda'|^2|w'|^2$$

That is, the space $\mathrm{Hom}_{\mathrm{HS}}(V, W)$ of Hilbert-Schmidt operators $V \to W$ is the *closure* of the space of finite-rank maps $V \to W$, in the space of all continuous linear maps $V \to W$, under the Hilbert-Schmidt norm. By construction, $\mathrm{Hom}_{\mathrm{HS}}(V, W)$ is a Hilbert space.

[1.4] Expressions for Hilbert-Schmidt norm, adjoints

The Hilbert-Schmidt norm of finite-rank $T : V \to W$ can be computed from any choice of orthonormal basis v_i for V, by

$$|T|^2_{\mathrm{HS}} \;=\; \sum_i |Tv_i|^2 \qquad\qquad \text{(at least for finite-rank } T)$$

Thus, taking a limit, the same formula computes the Hilbert-Schmidt norm of T known to be Hilbert-Schmidt. Similarly, for two Hilbert-Schmidt operators $S, T : V \to W$,

$$\langle S, T\rangle_{\mathrm{HS}} \;=\; \sum_i \langle Sv_i, Tv_i\rangle \qquad\qquad \text{(for any orthonormal basis } v_i)$$

The Hilbert-Schmidt norm $|\cdot|_{\mathrm{HS}}$ dominates the *uniform operator norm* $|\cdot|_{\mathrm{op}}$: given $\varepsilon > 0$, take $|v_1| \le 1$ with $|Tv_1|^2 + \varepsilon > |T|^2_o p$. Choose $v_2, v_3, \ldots$ so that $v_1, v_2, \ldots$ is an orthonormal basis. Then

$$|T|^2_{\mathrm{op}} \;\le\; |Tv_1|^2 + \varepsilon \;\le\; \varepsilon + \sum_n |Tv_n|^2 \;=\; \varepsilon + |T|^2_{\mathrm{HS}}$$

This holds for every $\varepsilon > 0$, so $|T|^2_{\mathrm{op}} \le |T|^2_{\mathrm{HS}}$. Thus, Hilbert-Schmidt limits are operator-norm limits, and Hilbert-Schmidt limits of finite-rank operators are *compact*.

Adjoints $T^* : W \to V$ of Hilbert-Schmidt operators $T : V \to W$ are Hilbert-Schmidt, since for an orthonormal basis w_j of W

$$\sum_i |Tv_i|^2 \;=\; \sum_{ij} |\langle Tv_i, w_j\rangle|^2 \;=\; \sum_{ij} |\langle v_i, T^* w_j\rangle|^2 \;=\; \sum_j |T^* w_j|^2$$

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[3] As usual a *finite-rank* linear map $T : V \to W$ is one with finite-dimensional image.

[4] Proof of this identification: on one hand, a map coming from $V^* \otimes_{\mathrm{alg}} W$ is a *finite* sum $\sum_i \lambda_i \otimes w_i$, so certainly has finite-dimensional image. On the other hand, given $T : V \to W$ with finite-dimensional image, take $v_1, \ldots, v_n$ be an orthonormal basis for the orthogonal complement $(\ker T)^\perp$ of $\ker T$. Define $\lambda_i \in V^*$ by $\lambda_i(v) = \langle v, v_i\rangle$. Then $T \sim \sum_i \lambda_i \otimes Tv_i$ is in $V^* \otimes W$. The second part of the argument uses the completeness of V.

[1.5] Criterion for Hilbert-Schmidt operators

We claim that a continuous linear map $T : V \to W$ with Hilbert space V is Hilbert-Schmidt if for some orthonormal basis v_i of V

$$\sum_i |Tv_i|^2 \;<\; \infty$$

and then (as above) that sum computes $|T|^2_{\text{HS}}$. Indeed, given that inequality, letting $\lambda_i(v) = \langle v, v_i\rangle$, T is Hilbert-Schmidt because it is the Hilbert-Schmidt limit of the finite-rank operators

$$T_n \;=\; \sum_{i=1}^n \lambda_i \otimes Tv_i$$

[1.6] Composition of Hilbert-Schmidt operators with continuous operators

Post-composing: for Hilbert-Schmidt $T : V \to W$ and continuous $S : W \to X$, the composite $S \circ T : V \to X$ is Hilbert-Schmidt, because for an orthonormal basis v_i of V,

$$\sum_i |S \circ Tv_i|^2 \;\le\; \sum_i |S|^2_{\text{op}} \cdot |Tv_i|^2 \;=\; |S|_{\text{op}} \cdot |T|^2_{\text{HS}} \qquad \text{(with operator norm } |S|_{\text{op}} = \sup_{|v|\le 1} |Sv|\text{)}$$

Pre-composing: for continuous $S : X \to V$ with Hilbert X and orthonormal basis x_j of X, since adjoints of Hilbert-Schmidt are Hilbert-Schmidt,

$$T \circ S \;=\; (S^* \circ T^*)^* \;=\; (\text{Hilbert-Schmidt})^* \;=\; \text{Hilbert-Schmidt}$$

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2. *Simplest nuclear Fréchet spaces*

Later, we will characterize a large class of *nuclear spaces*, a class of topological vector spaces behaving well with respect to tensor products in a categorical sense, aimed at a general Schwartz Kernel Theorem.

For the moment, we consider a special, more accessible, class of examples of nuclear spaces, sufficient for the Kernel Theorem for Sobolev spaces below.

[2.1] $V \otimes_{\text{HS}} W$ is not a categorical tensor product

Again, the Hilbert space $V \otimes_{\text{HS}} W$ is not a categorical tensor product of (infinite-dimensional) Hilbert spaces V, W. In particular, although the bilinear map $V \times W \to V \otimes_{\text{HS}} W$ is continuous, there are (jointly) continuous $\beta : V \times W \to X$ to Hilbert spaces H which do *not* factor through any continuous linear map $B : V \otimes_{\text{HS}} W \to X$.

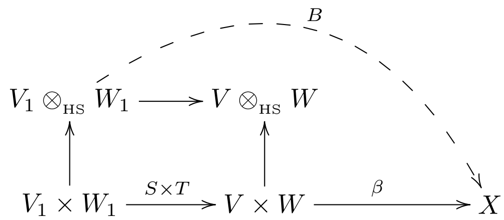
The case $W = V^*$ and $X = \mathbb{C}$, with $\beta(v, \lambda) = \lambda(v)$ already illustrates this point, since not every Hilbert-Schmidt operator has a trace. That is, letting v_i be an orthonormal basis for V and $\lambda_i(v) = \langle v, v_i\rangle$ an orthonormal basis for V^*, necessarily

$$B(\sum_{ij} c_{ij}\, v_i \otimes \lambda_j) \;=\; \sum_{ij} c_{ij}\beta(v_i, \lambda_j) \;=\; \sum_i c_{ii} \qquad (???)$$

However, $\sum_i \frac{1}{i} v_i \otimes \lambda_i$ is in $V \otimes_{\text{HS}} V^*$, but the alleged value of B is impossible. In other words, there are Hilbert-Schmidt maps which are not of trace class.

[2.2] Approaching tensor products and nuclear spaces

Let V, W, V_1, W_1 be Hilbert spaces with Hilbert-Schmidt maps $S : V_1 \to V$ and $T : W_1 \to W$. We claim that for any (jointly) continuous $\beta : V \times W \to X$, there is a unique continuous $B : V_1 \otimes_{\rm HS} W_1 \to X$ giving a commutative diagram



In fact, $B : V_1 \otimes_{\rm HS} W_1 \to X$ is *Hilbert-Schmidt*. As the diagram suggests, $V \otimes_{\rm HS} W$ is bypassed, playing no role.

Proof: Once the assertion is formulated, the argument is the only thing it can be: The continuity of β gives a constant C such that $|\beta(v, w)| \le C \cdot |v| \cdot |w|$, for all $v \in V$, $w \in W$. The Hilbert-Schmidt condition is that, for chosen orthonormal bases v_i of V_1 and w_j of W_1,

$$|S|^2_{\rm HS} \;=\; \sum_i |Sv_i|^2 \;<\; \infty \qquad\qquad |T|^2_{\rm HS} \;=\; \sum_j |Tw_i|^2 \;<\; \infty$$

Thus,

$$|\beta(Sv, Tw)| \;\le\; C \cdot |Sv| \cdot |Tv|$$

Squaring and summing over v_i and w_j,

$$\sum_{ij} |\beta(Sv_i, Tw_j)|^2 \;\le\; C \cdot \sum_{ij} |Sv_i|^2 \cdot |Tw_j|^2 \;=\; C \cdot |S|^2_{\rm HS} \cdot |T|^2_{\rm HS} \;<\; \infty$$

That is, the obvious definition-attempt

$$B(\sum_{ij} c_{ij}\, v_i \otimes w_j) \;=\; \sum_{ij} c_{ij}\, \beta(Sv_i, Tw_j)$$

does produce a Hilbert-Schmidt operator $V_1 \otimes W_1 \to X$. ///

[2.3] A class of nuclear Fréchet spaces

We take the basic *nuclear Fréchet space* to be a countable limit [5] of Hilbert spaces where the transition maps are *Hilbert-Schmidt*.

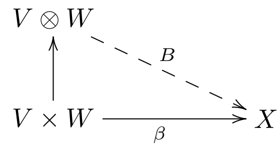
That is, for a countable collection of Hilbert spaces $V_0, V_1, V_2, \ldots$ with *Hilbert-Schmidt* maps $\varphi_i : V_i \to V_{i-1}$, the limit $V = \lim_i V_i$ in the category of locally convex topological vector spaces is a *nuclear Fréchet space.* [6]

Let $\mathfrak{C}$ be the category of Hilbert spaces enlarged to include limits.

[5] Properly, the class of categorical *limits* includes *products* and other objects whose indexing sets are not necessarily *directed.* In that context, requiring that the index set be directed, a projective limit is a *directed* or *filtered* limit. Similarly, what we will call simply *colimits* are properly *filtered* or *directed* colimits.

[6] The new aspect is the nuclearity, not the Fréchet-ness: an arbitrary *countable* limit of Hilbert spaces is (provably) Fréchet, since an arbitrary countable limit of *Fréchet* spaces is Fréchet.

We claim that *nuclear Fréchet spaces admit tensor products* in $\mathfrak{C}$. That is, for nuclear spaces $V = \lim_i V_i$ and $W = \lim W_i$ there is a nuclear space $V \otimes W$ and continuous bilinear $V \otimes W \to V \otimes W$ such that, given a jointly continuous bilinear map $\beta : V \times W \to X$ of nuclear spaces V, W to $X \in \mathfrak{C}$, there is a unique continuous linear map $B : V \otimes W \to X$ giving a commutative diagram



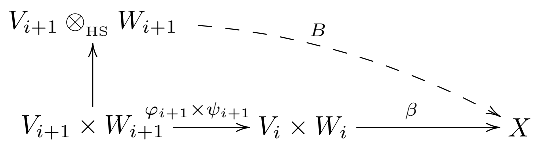
In particular, $V \otimes W \approx \lim_i V_i \otimes_{\text{HS}} W_i$.

Proof: By the defining property of (projective) limits, it suffices to treat the case that X is itself a Hilbert space. Let $\varphi_i : V_i \to V_{i-1}$ and $\psi_i : W_i \to W_{i-1}$ be the transition maps. First, we claim that, for large-enough index i, the bilinear map $\beta : V \times W \to X$ factors through $V_i \times W_i$. Indeed, the topologies on V and W are such that, given $\varepsilon_o > 0$, there are indices i, j and open neighborhoods of zero $E \subset V_i$, $F \subset W_j$ such that $\beta(E \times F) \subset \varepsilon_o$-ball at 0 in X. Since β is $\mathbb{C}$-bilinear, for *any* $\varepsilon > 0$,

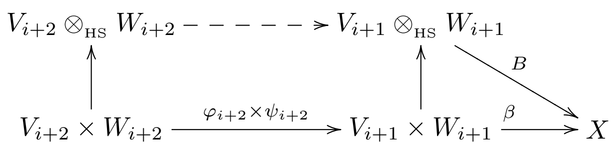
$$\beta(\frac{\varepsilon}{\varepsilon_o} E \times F) \;\subset\; \varepsilon\text{-ball at 0 in } X$$

That is, β is already continuous in the $V_i \times W_j$ topology. Replace i, j by their maximum, so $i = j$.

The argument of the previous section exhibits continuous linear B fitting into the diagram



In fact, B is Hilbert-Schmidt. Applying the same argument with X replaced by $V_{i+1} \otimes_{\text{HS}} W_{i+1}$ shows that the dotted map in



is Hilbert-Schmidt. Thus, the categorical tensor product is the limit of the Hilbert-Schmidt completions of the algebraic tensor products of the limitands:

$$(\lim_i V_i) \otimes (\lim_j W_j) \;=\; \lim_i \big(V_i \otimes_{\text{HS}} W_i\big)$$

The transition maps in this limit have been proven Hilbert-Schmidt, so the limit is again nuclear. ///

[2.4] Example: tensor products of Sobolev spaces

Let $\mathbb{T}$ be the circle $\mathbb{R}/2\pi\mathbb{Z}$. In terms of Fourier series, for $s \geq 0$ the s^{th} L^2 Sobolev space on $\mathbb{T}^m$ is

$$\text{Sob}(s, \mathbb{T}^m) \;=\; \{\sum_\xi c_\xi \, e^{i\xi \cdot x} \in L^2(\mathbb{T}^m) \;:\; \sum_\xi |c_\xi|^2 \cdot (1 + |\xi|^2)^s < \infty\}$$

The Sobolev imbedding theorem asserts that

$$\text{Sob}(k + \tfrac{m}{2} + \varepsilon, \mathbb{T}^m) \;\subset\; C^k(\mathbb{T}^m) \qquad\qquad \text{(for all } \varepsilon > 0)$$

Thus,

$$C^\infty(\mathbb{T}^m) \;=\; \mathrm{Sob}(+\infty, \mathbb{T}^m) \;=\; \lim_s \mathrm{Sob}(s, \mathbb{T}^m) \;\approx\; \lim \Big(\ldots \to \mathrm{Sob}(2, \mathbb{T}^m) \to \mathrm{Sob}(1, \mathbb{T}^m) \to \mathrm{Sob}(0, \mathbb{T}^m) \Big)$$

We claim that

$$\mathrm{Sob}(+\infty, \mathbb{T}^m) \otimes_{\mathfrak{C}} \mathrm{Sob}(+\infty, \mathbb{T}^n) \;\approx\; \mathrm{Sob}(+\infty, \mathbb{T}^{m+n})$$

induced from the natural

$$(\varphi \otimes \psi)(x, y) \;=\; \varphi(x)\, \psi(y) \qquad\qquad (\varphi \in \mathrm{Sob}(+\infty, \mathbb{T}^m),\ \psi \in \mathrm{Sob}(+\infty, \mathbb{T}^n),\ x \in \mathbb{T}^m,\ y \in \mathbb{T}^n)$$

Indeed, our construction of this tensor product is

$$\mathrm{Sob}(+\infty, \mathbb{T}^m) \otimes_{\mathfrak{C}} \mathrm{Sob}(+\infty, \mathbb{T}^n) \;=\; \lim_s \Big(\mathrm{Sob}(s, \mathbb{T}^m) \otimes_{\mathrm{HS}} \mathrm{Sob}(s, \mathbb{T}^n) \Big)$$

The inequalities

$$(1 + |\xi|^2 + |\eta|^2)^2 \;\geq\; (1 + |\xi|^2)(1 + |\eta|^2) \;\geq\; 1 + |\xi|^2 + |\eta|^2 \qquad\qquad (\text{for } \xi \in \mathbb{Z}^m,\ \eta \in \mathbb{Z}^n)$$

give

$$\mathrm{Sob}(2s, \mathbb{T}^{m+n}) \;\subset\; \mathrm{Sob}(s, \mathbb{T}^m) \otimes_{\mathrm{HS}} \mathrm{Sob}(s, \mathbb{T}^n) \;\subset\; \mathrm{Sob}(s, \mathbb{T}^{m+n}) \qquad\qquad (\text{for } s \geq 0)$$

The limit only depends on cofinal subsystems, so, indeed,

$$\mathrm{Sob}(+\infty, \mathbb{T}^m) \otimes_{\mathfrak{C}} \mathrm{Sob}(+\infty, \mathbb{T}^n) \;\approx\; \mathrm{Sob}(+\infty, \mathbb{T}^{m+n})$$

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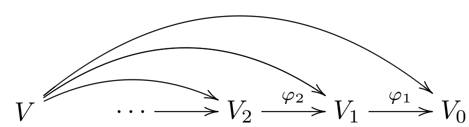
3. *Strong dual topologies and colimits*

The example of the *Schwartz Kernel Theorem* below refers to *duals* of Sobolev spaces, so the nature of the topology must be made clear.

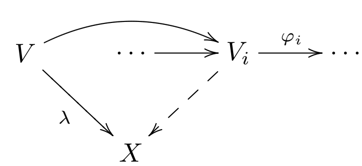
[3.1] Duals of limits of Banach spaces

The topology on a limit



of Banach spaces V_i is given by the norms $|\cdot|_i$ on V_i, composed with the maps $\sigma_i : V \to V_i$, giving seminorms $p_i = |\cdot|_i \circ \sigma_i$.

We claim that linear maps $\lambda : V \to X$ from $V = \lim_i V_i$ of Banach spaces V_i to a normed space X necessarily factor through some limitand:



Proof: Without loss of generality, replace each V_i by the closure of the image of V_i in it. Continuity of λ is that, given $\varepsilon > 0$, there is an index i and a $\delta > 0$ such that

$$\lambda\Big(\{ v \in V : p_i(v) < \delta \} \Big) \;\subset\; \{ x \in X : |x|_X < \varepsilon \}$$

Then, for any $\varepsilon' > 0$,

$$\lambda\Big(\{v \in V : p_i(v) < \delta \cdot \frac{\varepsilon'}{\varepsilon}\}\Big) \;\subset\; \{x \in X : |x|_X < \varepsilon'\}$$

Thus, λ extends by continuity to the closure of $\sigma_i V$ in V_i, and gives a continuous map $V_i \to X$. ///

Thus, the *dual* of a limit of Banach spaces V_i is a colimit

$$V_0^* \xrightarrow{\varphi_1^*} V_1^* \xrightarrow{\varphi_1^*} V_2^* \longrightarrow \cdots \qquad \operatorname{colim} V_i^*$$

The duals V_i^* and the colimit are unambiguous as vector spaces. The topology on the colimit depends on the choice of topology on the duals V_i^*.

One reason to consider limits of Banach spaces V_i is the natural Banach-space structure on the dual. These are examples of *strong dual* topologies. In general, the *strong dual* topology on the dual V^* of a locally convex topological vector space V is given by seminorms

$$p_E(\lambda) \;=\; \sup_{v \in E} |\lambda v| \qquad (E \text{ a } \textit{bounded}\text{, convex, balanced neighborhood of 0 in } V)$$

Recall that a *bounded* set E in a general topological vector space V is characterized by the property that, for every open neighborhood U of 0 in V, there is t_o such that $tU \supset E$ for all $t \geq t_o$.

Consider a countable limit $V = \lim V_i$ of Banach spaces, where for simplicity we suppose that all transition maps $V_i \to V_{i-1}$ are *injections*. We claim that the (locally convex) colimit $\operatorname{colim}_i(V_i^*)$ of the strong duals V_i^* gives the *strong dual* topology on the dual V^* of the limit $V = \lim V_i$.

Proof: Since the transition maps $V_i \to V_{i-1}$ are injections, as a set the limit V is the nested intersection of the V_i, and we identify V_i as a subset of V_{i-1}. Further, the dual V^* is identifiable with the ascending union of the duals V_i^*, regardless of topology.

The first point is to show that every bounded subset of V is contained in a bounded subset E expressible as a nested intersection of bounded subsets E_i of V_i. To see this, first note that the topology on V is given by the collection of (semi-) norms $|\cdot|_i$ on the individual Banach spaces V_i. A set $E \subset$ is bounded if and only if, for every index i, there is a radius r_i such that E is inside the ball $B_i(r_i)$ of radius r_i in V_i. We may as well replace these balls by the intersection of all the lower-(or-equal-)index balls:

$$E_i \;=\; \bigcap_{j \geq i} B_j(r_j)$$

The set E_i is bounded in V_i, $E_i \subset E_{i-1}$, and E is their nested intersection.

Now consider the linear functionals. On one hand, a given $\lambda : V \to \mathbb{C}$ factors through some $\lambda_i \in V_i^*$, and λE being inside the ε-ball B_ε in $\mathbb{C}$ is implied by $\lambda_i E_i \subset B_\varepsilon$ for some i. On the other hand, for $\lambda E \subset B_\varepsilon$, we claim $\lambda E_i \subset B_\varepsilon$ for large-enough i. Indeed, λE_i is a balanced, bounded, convex subset of $\mathbb{C}$, so is a disk (either open or closed) of radius r_i. Since the intersection of the λE_i is inside B_ε, necessarily $\lim r_i \leq \varepsilon$, with strict inequality if the disks are closed. Thus, there is i_o such that $r_i \leq \varepsilon$ for $i \geq i_o$, with $r_i < \varepsilon$ for close disks. Thus, there is i_o such that $\lambda E_i \subset B_\varepsilon$ for $i \geq i_o$.

That is, the strong dual topology on $V^* = \bigcup_i V_i^*$ is the colimit of the strong dual (Banach) topologies on the V_i^*. ///

[3.1.1] Remark: The locally convex colimit of the Hilbert spaces $\mathrm{Sob}(-s, \mathbb{T}^n)$ is denoted $\mathrm{Sob}(-\infty, \mathbb{T}^n)$, especially after verifying that the colimit topology from the strong duals $\mathrm{Sob}(-s, \mathbb{T}^n)$ is the strong dual topology on $\mathrm{Sob}(+\infty, \mathbb{T}^n)^*$.

4. *Schwartz Kernel Theorem for Sobolev spaces*

Continue the example of Sobolev spaces on products $\mathbb{T}^m$ of circles $\mathbb{T}$. The following is the simplest example of Schwartz' Kernel Theorem:

We claim that the map

$$\mathrm{Hom}^o\big(\mathrm{Sob}(\infty,\mathbb{T}^m),\, \mathrm{Sob}(\infty,\mathbb{T}^n)^*\big) \;\approx\; \mathrm{Sob}(\infty,\mathbb{T}^{m+n})^*$$

induced by

$$\big(f \longrightarrow (F\to\Phi(f\otimes F)\big) \quad\longleftarrow\quad \Phi \qquad (f\in\mathrm{Sob}(\infty,\mathbb{T}^m),\, F\in\mathrm{Sob}(\infty,\mathbb{T}^n),\, \Phi\in\mathrm{Sob}(\infty,T^{m+n})^*)$$

is an isomorphism.

[4.0.1] **Remark:** The Hom-space Hom^o is *continuous* linear maps, so giving sense to the assertion requires a topology on the dual space $\mathrm{Sob}(\infty,\mathbb{T}^n)^*$. Most optimistically, because this would most-constrain the continuous maps, we give this dual the *strong dual* topology $\mathrm{Sob}(-\infty,\mathbb{T}^n)$.

[4.0.2] **Remark:** The distribution $\Phi\in\mathrm{Sob}(\infty,\mathbb{T}^{m+n})^*$ producing a given continuous map from $\mathrm{Sob}(\infty,\mathbb{T}^m)$ to $\mathrm{Sob}(\infty,\mathbb{T}^n)^*$ is the *Schwartz kernel* of the map.

Proof: Let $X=\mathrm{Sob}(\infty,\mathbb{T}^m)$ and $Y=\mathrm{Sob}(\infty,T^n)$, Given the existence of the categorical tensor product, established above, it suffices to show that the vector space

$$\mathrm{Bil}^o(X\times Y,\mathbb{C})$$

of jointly continuous bilinear maps is linearly isomorphic to $\mathrm{Hom}(X,Y^*)$, via the expected

$$\beta \;\longrightarrow\; (x\longrightarrow(y\to\beta(x,y))) \qquad (\text{for } \beta\in\mathrm{Bil}^o(X,Y),\ x\in X, \text{ and } y\in Y)$$

where Y^* is given the *strong dual* topology. It is immediate that the map is a bijection. The issue is only topological.

Given $x\in X$, bounded $E\subset Y$, and $\varepsilon>0$, by joint continuity of β, there are neighborhoods M,N of 0 in X,Y such that

$$\beta(x+M,N) \;=\; \beta(x+M,N)-\beta(x,0) \;\subset\; \varepsilon\text{-ball in } Y^*$$

Since E is bounded, there is $t>0$ such that $tN\supset E$. Then

$$\beta(x+m,e)-\beta(x,e) \;=\; \beta(m,e) \;\in\; \beta(M,E) \;\subset\; \beta(M,tN) \qquad (\text{for } m\in M \text{ and } e\in E)$$

This suggests replacing M by $t^{-1}M$, so

$$\beta(x+m,e)-\beta(x,e) \;=\; \beta(t^{-1}M,E) \;\subset\; \beta(t^{-1}M,tN) \;\subset\; \varepsilon\text{-ball in } Y^* \qquad (\text{for } m\in t^{-1}M \text{ and } e\in E)$$

That is,

$$\beta(x+m,-)-\beta(x,-) \;\in\; U_{E,\varepsilon} \qquad (\text{for } m\in t^{-1}M)$$

This proves the continuity of the map $X\to Y^*$ induced by β.

Conversely, given $\varphi: X\to Y^*$, put $\beta(x,y)=\varphi(x)(y)$. For fixed x, $\beta(x,-)=\varphi(x)$ is continuous, by hypothesis. For fixed y, $E=\{y\}$ is a bounded set in Y, so by the continuity of $x\to\varphi(x)$, for given x and $\varepsilon>0$ there is a neighborhood M of 0 in X so that $\varphi(x+M)-\varphi(x)\subset U_{E,\varepsilon}$. This proves that $\beta(-,y)$ is continuous. Thus, β is *separately* continuous. Since X and Y are *Fréchet*, separately continuous bilinear functions are jointly continuous. ///

5. *Appendix: joint continuity of bilinear maps on Fréchet spaces*

One of the family of corollaries of Baire category and related ideas is the following convenient standard fact:

Let $\beta : X \times Y \to Z$ be a bilinear map on Fréchet spaces X, Y, continuous in each variable separately. Then β is *jointly* continuous.

Proof: Fix a neighborhood N of 0 in Z Let $x_n \to x_o$ in X and $y_n \to y_o$ in Y. For each $x \in X$, by continuity in Y, $\beta(x, y_n) \to \beta(x, y_o)$. Thus, for each $x \in X$, the set of values $\beta(x, y_n)$ is *bounded* in Z. The linear functionals $x \to \beta(x, y_n)$ are *equicontinuous*, by Banach-Steinhaus, so there is a neighborhood U of 0 in X so that $b_n(U) \subset N$ for all n. In the identity

$$\beta(x_n, y_n) - \beta(x_o, y_o) \;=\; \beta(x_n - x_o, y_n) + \beta(x_o, y_n - y_o)$$

we have $x_n - x_o \in U$ for large n, and $\beta(x_n - x_o, y_o) \in N$. Also, by continuity in Y, $\beta(x_o, y_n - y_o) \in N$ for large n. Thus, $\beta(x_n, y_n) - \beta(x_o, y_o) \in N + N$, proving *sequential* continuity. Since $X \times Y$ is metrizable, sequential continuity implies continuity. ///

[5.0.1] Remark: The roles of X, Y in the argument are somewhat unsymmetrical, suggesting technical sharpening of the assertion, but we do not need that.

Bibliography

[Garrett 2010] P. Garrett, *Non-existence of tensor products of Hilbert spaces*, http://www.math.umn.edu/˜garrett/m/v/nonexistence_tensors.pdf

[Gelfand-Silov 1964] I.M. Gelfand, G.E. Silov, *Generalized Functions, I: Properties and Operators*, Academic Press, NY, 1964.

[Grothendieck 1955] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Am. Math. Soc. **16**, 1955.

[Schwartz 1950/51] L. Schwartz, *Théorie des Distributions*, I,II Hermann, Paris, 1950/51, 3rd edition, 1965.

[Schwartz 1950] L. Schwartz, *Théorie des noyaux*, Proc. Int. Cong. Math. Cambridge 1950, I, 220-230.

[Schwartz 1953/4] L. Schwartz, *Espaces de fonctions différentiables à valeurs vectorielles*, J. d'Analyse Math. **4** (1953/4), 88-148.

[Schwartz 1953/54b] L. Schwartz, *Produit tensoriels topologiques*, Séminaire, Paris, 1953-54.

[Schwartz 1957/59] L. Schwartz, *Distributions à valeurs vectorielles, I, II*, Ann. Inst. Fourier Grenoble **VII** (1957), 1-141, **VIII** (1959), 1-207.

[Taylor 1995] J.L. Taylor, *Notes on locally convex topological vector spaces*, course notes from 1994-95, http://www.math.utah.edu/˜taylor/LCS.pdf

[Trèves 1967] F. Trèves, *Topological vector spaces, distributions, and kernels*, Academic Press, 1967, reprinted by Dover, 2006.