Poisson Summation by Distribution-Theory

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1 Classical Poisson summation

Let $\psi(y) = e^{2\pi i y}$ and $\psi_x(y) = \psi(xy)$. The classical **Fourier transform** of an L^1-function f on $\mathbf{R}$ is

$$\mathcal{F}f(x) = \hat{f}(x) := \int_{\mathbf{R}} f(y)\,\psi(-xy)\,dy$$

The space of **Schwartz functions** $\mathcal{S} = \mathcal{S}(\mathbf{R})$ on $\mathbf{R}$ is the space of smooth (i.e., infinitely-differentiable) functions f on $\mathbf{R}$ so that for all non-negative integers m, n the value

$$\nu_{m,n}(f) := \sup_x (1+x^2)^m\,|f^{(n)}(x)|$$

is *finite*. The functions $\nu_{m,n}$ form a countable collection of seminorms, with respect to which $\mathcal{S}$ is *complete*. Thus, $\mathcal{S}$ is a (locally convex, separable) *Frechet space*.

The space of **test functions** $\mathcal{D}(\mathbf{R}) = C_c^\infty(\mathbf{R})$ on $\mathbf{R}$ is the direct limit

$$C_c^\infty(\mathbf{R}) := \text{ dir.lim }_K\, C_c^\infty(K) = \bigcup_K C_c^\infty(K)$$

where K ranges over compact subsets of $\mathbf{R}$, and the maps in the direct limit are the inclusions. Each space $C_c^\infty(K)$ has a (locally convex, separable) Frechet space structure given by the countable collection of seminorms

$$\mu_n(f) := \sup_{x \in K} |f^{(n)}(x)|$$

The inclusion maps are certainly continuous, and make $C_c^\infty(K)$ a closed subspace of $C_c^\infty(K')$ for $K \subset K'$.

The space of **distributions** on $\mathbf{R}$ is the continuous dual $\mathcal{D}' = C_c^\infty(\mathbf{R})'$ of the space of test functions $\mathcal{D} := C_c^\infty(\mathbf{R})$. The space of **tempered distributions** on $\mathbf{R}$ is the continuous dual $\mathcal{S}'$ of the space of Schwartz functions.

The space of all distributions is a module over the ring $C_o^\infty(\mathbf{R})$ of *all* smooth functions in the following way: for $\varphi \in C_o^\infty(\mathbf{R})$, for a distribution u, and for a test function f, define

$$(\varphi u)(f) := u(\varphi f)$$

where φf is the usual pointwise product.

One can check that the natural inclusion $C_c^\infty(\mathbf{R}) \to \mathcal{S}(\mathbf{R})$ is *continuous*, and that the image is dense. Thus, the inclusion $\mathcal{D} \to \mathcal{S}$ induces a map $\mathcal{S}' \to \mathcal{D}'$

which is continuous in the weak star-topologies on these dual spaces. That is, every tempered distribution is a 'plain' distribution: this is useful in some situations where proof of uniqueness or non-existence of some sort of *tempered* distribution proceeds most reasonably by proving the formally stronger assertion about *'plain'* distributions.

The Fourier transform maps $\mathcal{S}$ *continuously* to itself, with inverse given by the inverse transform

$$\check{f}(x) := \int_{\mathbf{R}} f(y)\,\psi(xy)\,dy$$

Therefore, for a tempered distribution u we can define a Fourier transform by

$$(\mathcal{F}u)(f)\hat{u}(f) := u(\hat{f})$$

Note that the space of test functions is *not* mapped to itself by Fourier transform (compact support is not preserved), so we cannot reasonably define a Fourier transform on 'plain' distributions.

This is a suitable generalization of Fourier transform of a Schwartz function, in the following sense. To a Schwartz function φ we associate a distribution u_φ defined by

$$u_\varphi(f) := \int_{\mathbf{R}} \varphi(x)\,f(x)\,dx$$

The asserted compatibility is that

$$\mathcal{F}(u_\varphi) = u_{\mathcal{F}\varphi}$$

This follows from the equality

$$\int \hat{\varphi}(x)\,f(x)\,dx = \int \varphi(x)\hat{f}(x)\,dx$$

The **support** $\mathrm{spt}(u)$ of a distribution u is the smallest closed set C so that for $f \in C_c^\infty(\mathbf{R})$ with $\mathrm{spt}(f) \cap C = \varphi$ we have $u(f) = 0$. We claim that the distributions with support consisting of a single point $\{x_0\}$ are the finite linear combinations of (distributional) derivatives $\delta_{x_0}^{(n)}$ of the Dirac delta function δ_{x_0} at x_0. (The latter is defined by

$$\delta_{x_0}(f) := f(x_0)$$

and also

$$\delta_{x_0}^{(n)}(f) := f^{(n)}(x_0)$$

are the derivatives.) *Proof: omitted for now*

The additive group $\mathbf{R}$ acts on $\mathcal{S}$ and on $\mathcal{D}$ by the **regular representation**

$$R_g f(x) := f(x+g)$$

This action is continuous. The natural duality gives the (continuous) **dual** or **contragredient** representation on $\mathcal{S}'$ and $\mathcal{D}'$ by

$$(R'_g u)(f) := u(R_{g^{-1}} f)$$

We have two fundamental identities regarding this regular representation and Fourier transforms (for $f \in \mathcal{S}$):

$$(R_x f)\hat{} = \psi_x \hat{f}$$

$$(\psi_x f) = R_{-x}(\hat{f})$$

(by direct computation). From these and from the definition of Fourier transform for tempered distributions, the same identities must hold for tempered distributions, as well.

Now let Φ be a collection of smooth functions on $\mathbf{R}$ having *common zero set* Z. Let u be a distribution such that $\varphi u = 0$ for all $\varphi \in \Phi$. We claim that

$$\mathrm{spt}(u) \subset Z$$

Proof: omitted for now

Refining a special case of the previous result, suppose that Φ is a subset of $C_o^\infty(\mathbf{R})$ having a single point as common zero set. Without loss of generality, we suppose that this single point is 0. Let $\mathcal{O}_0$ be the ring of germs of smooth functions at 0, and let $\mathbf{m}$ be its unique maximal ideal, consisting of smooth functions vanishing at 0. Suppose that the image in $\mathcal{O}_0$ of the ideal generated by Φ in $C_o^\infty(\mathbf{R})$ is $\mathbf{m}^n$. That is, we suppose that all functions in Φ vanish at 0 to order at least n, and every germ of a smooth function at 0 vanishing to order at least n is a linear combination over $\mathcal{O}_0$ of elements of Φ. Let u be a distribution so that $\varphi u = 0$ for all $\varphi \in \Phi$. *Then u is a complex linear combination of $\delta_0, \ldots, \delta_0^{(n-1)}$. Proof: omitted for now*

Now consider the two tempered distributions

$$u(f) := \sum_{n \in \mathbf{Z}} f(n)$$

$$v(f) := \sum_{n \in \mathbf{Z}} \hat{f}(n)$$

The *Poisson summation formula* asserts that

$$u = v$$

We will isolate properties possessed by both u and v, and then prove that there is a unique tempered distribution with these properties.

Certainly $u(\psi_n f) = u(f)$ for $n \in \mathbf{Z}$, and $u(R_n f) = u(f)$ for $n \in \mathbf{Z}$. Thus,

$$\psi_n u = u \qquad R_n u = u \qquad \text{for all } n \in \mathbf{Z}.$$

The two identities above which 'intertwine' the Fourier transform and the regular representation imply that v has the same properties. Further, letting $\gamma(x) := e^{-\pi x^2}$, we have $\hat{\gamma} = \gamma$, and so

$$u(\gamma) = v(\gamma)$$

Now we prove that the space of tempered distributions w such that

$$\psi_n w = w \qquad R_n w = w \qquad \text{for all } n \in \mathbf{Z}.$$

is *one-dimensional* over $\mathbf{C}$. This, together with the evaluation of both u and v on γ, will prove the Poisson summation formula. The common zero set of the collection

$$\Phi = \{\psi_n - 1 : n \in \mathbf{Z}\}$$

is $\mathbf{Z}$, so any 'plain' distribution w annihilated by multiplication by all $\psi_n - 1$ must be supported on $\mathbf{Z}$. Let $\varphi \in C_o^\infty(\mathbf{R})$ be such that $\text{spt}(\varphi) \cap \mathbf{Z} = \{0\}$ and $\varphi \equiv 1$ on some neighborhood of 0. Then $\text{spt}(\varphi w) = 0$. Further, since the $\psi_n - 1$ generate the whole maximal ideal in the ring of germs of smooth functions at 0, we conclude that φw is a constant multiple of δ_0. By use of a partition of unity to '*localize*' the issue, we find that

$$w = \sum c_n \, \delta_n$$

for some constants c_n. The 'translation invariance' of w implies that all the constants c_n must be the same. Thus, there is a constant c so that

$$w = c \sum \delta_n$$

This is the desired uniqueness (one-dimensionality) assertion.