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Riesz' Lemma

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This elementary inequality is often referred-to as *Riesz' Inequality*, but is much less often indexed as such in the relevant texts and monographs, making it inconvenient to find by that name. It is simple and widely useful. A sample reference is page 218, [Riesz-Nagy 1952].

This little lemma is the Banach-space substitute for one aspect of *orthogonality* in Hilbert spaces. In a Hilbert space Y, given a non-dense subspace X, there is $y \in Y$ with $|y| = 1$ and $\inf_{x\in X} |x-y| = 1$, by taking y in the *orthogonal complement* to X. Of course, this device is unavailable more generally.

[0.0.1] Lemma: For a non-dense subspace X of a Banach space Y, given $0 < r < 1$, there is $y \in Y$ with $|y| = 1$ but $\inf_{x\in X} |x-y| \geq r$.

Proof: Take y_1 not in the closure of X, and put $R = \inf_{x\in X} |x - y_1|$. Thus, $R > 0$. For $\varepsilon > 0$, let $x_1 \in X$ be such that $|x_1 - y_1| < R + \varepsilon$. Put $y = (y_1 - x_1)/|x_1 - y_1|$, so $|y| = 1$. And

$$\inf_{x\in X} |x-y| \;=\; \inf_{x\in X} |x + \frac{x_1}{|x_1-y_1|} - \frac{y_1}{|x_1-y_1|}| \;=\; \frac{\inf_{x\in X} |x-y_1|}{|x_1-y_1|} \;=\; \frac{R}{R+\varepsilon}$$

By choosing $\varepsilon > 0$ small, $R/(R+\varepsilon)$ can be made arbitrarily close to 1. ///

Bibliography

[Riesz-Nagy 1952] F. Riesz, B. Szökefalvi.-Nagy, *Functional Analysis*, English translation, 1955, L. Boron, from *Lecons d'analyse fonctionelle* 1952, F. Ungar, New York, 1955
