(*March 20, 2012*)

Simplest Sobolev imbedding and Rellich-Kondrachev compactness

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We prove the simplest case of a Sobolev imbedding theorem, namely, that the +1-index Sobolev space $H^1[0,1]$ (below) is inside $C^o[0,1]$.

Second, we prove the simplest case of Rellich-Kondrachev compactness lemma, that the inclusion $H^1[0,1] \subset L^2[0,1]$ is *compact*.

All that is used is the fundamental theorem of calculus, Cauchy-Schwarz-Bunyakowsky inequality, and the total boundedness criterion for a set's having compact closure in a complete metric space.

These are old results, and have been much elaborated in the intervening decades. However, the underlying causal mechanisms are simple and fundamental.

Let

$$L^2[a,b] \;=\; \text{completion of } \; C_c^\infty[a,b] \text{ with respect to } |f| = |f|_{L^2} = \Big(\int_a^b |f(t)|^2\,dt\Big)^{1/2}$$

The +1-index Sobolev space [1] $H^1[a,b]$ is

$$H^1[a,b] \;=\; \text{completion of } \; C_c^\infty[a,b] \text{ with respect to } |f|_{H^1} = \Big(|f|^2 + |f'|^2\Big)^{1/2}$$

[0.0.1] Theorem: *(Sobolev imbedding)* $H^1[0,1] \subset C^o[0,1]$.

[0.0.2] Theorem: *(Rellich-Kondrachev)* The inclusion $H^1[0,1] \to L^2[0,1]$ is *compact*.

Proof: (of Sobolev imbedding) Prove that the H^1 norm dominates the C^o norm, namely, sup-norm, on $C_c^\infty[0,1]$. First, for $0 \le x \le y \le 1$, the difference between maximum and minimum values of $f \in C_c^\infty[0,1]$ is constrained:

$$|f(y) - f(x)| \;=\; \Big|\int_x^y f'(t)\,dt\Big| \;\le\; \int_0^1 |f'(t)|\,dt \;\le\; \Big(\int_0^1 |f'(t)|^2\,dt\Big)^{1/2} \cdot \Big(\int_x^y 1\,dt\Big)^{1/2} \;=\; |f'|_{L^2} \cdot |x-y|^{\frac{1}{2}}$$

Let $y \in [0,1]$ be such that $|f(y)| = \min_x |f(x)|$. Then, using this inequality,

$$|f(x)| \;\le\; |f(y)| + |f(x) - f(y)| \;\le\; \int_0^1 |f(t)|\,dt + |f(x) - f(y)|$$

$$\le\; \int_0^1 |f| \cdot 1 + |f'|_{L^2} \cdot 1 \;\le\; |f|_{L^2} + |f'|_{L^2} \;\ll\; 2(|f|^2 + |f'|^2)^{1/2} \;=\; 2|f|_{H^1}$$

Thus, on $C_c^\infty[0,1]$ the H^1 norm dominates the sup-norm. Thus, this comparison holds on the H^1 completion $H^1[0,1]$, and $H^1[0,1] \subset C^o[0,1]$. ///

[0.0.3] Corollary: (of proof) $|f(x) - f(y)| \le |f'|_{L^2} \cdot |x-y|^{\frac{1}{2}}$ for $f \in H^1[0,1]$. ///

[1] ... also denoted $W^{1,2}[a,b]$, where the superscript 2 refers to L^2, rather than L^p.

Proof: (of Rellich-Kondrachev) Prove that the unit ball in $H^1[0,1]$ is *totally bounded* in $L^2[0,1]$. Approximate $f \in H^1[0,1]$ in $L^2[0,1]$ by piecewise-constant functions

$$F(x) \;=\; \begin{cases} c_1 & \text{for } 0 \le x < \frac{1}{n} \\ c_2 & \text{for } \frac{1}{n} \le x < \frac{2}{n} \\ \ldots & \\ c_n & \text{for } \frac{n-1}{n} \le x \le 1 \end{cases}$$

From above, for $|f|_{H^1} \le 1$, the sup norm is bounded by 2, so we need only consider c_i in the range $|c_i| \le 2$. Since $|f(x) - f(y)| \ll |x-y|^{\frac{1}{2}}$ for $|f|_{H^1} \le 1$,

Given $\varepsilon > 0$, take N large enough such that the disk of radius 2 in $\mathbb{C}$ is covered by N disks of radius less than ε, with centers C. Given $f \in H^1[0,1]$ with $|f|_1 \le 1$, choose constants $c_k \in C$ such that $|f(k/n) - c_k| < \varepsilon$. Then

$$|f(x) - c_k| \;\le\; |f\Big(\frac{k}{n}\Big) - c_k| + \Big|f(x) - f\Big(\frac{k}{n}\Big)\Big| \;<\; \varepsilon + \Big|x - \frac{k}{n}\Big|^{\frac{1}{2}} \;\le\; \varepsilon + \frac{1}{\sqrt{n}} \qquad (\text{for } \tfrac{k}{n} \le x \le \tfrac{k+1}{n})$$

Then

$$\int_0^1 |f - F|^2 \;\le\; \sum_{k=1}^n \int_{k/n}^{(k+1)/n} \Big(\varepsilon + \frac{1}{\sqrt{n}}\Big)^2 \;\le\; n \cdot \frac{1}{n} \cdot \Big(\varepsilon + \frac{1}{\sqrt{n}}\Big)^2 \;=\; \Big(\varepsilon + \frac{1}{\sqrt{n}}\Big)^2$$

For ε small and n large, this is small. Thus, the image in $L^2[0,1]$ of the unit ball in $H^1[0,1]$ is totally bounded, so has compact closure. This proves that the inclusion $H^1[0,1] \subset L^2[0,1]$ is *compact.* ///

[0.0.4] Remark: The one-dimensional L^2 case treated here is much simpler than the general case $1 \le p < +\infty$.
