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Sturm-Liouville alternation-of-roots phenomenon

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Here we prove a funny little result, that a 'higher' eigenfunction (for a standard kind of second-order differential operator) has a 0 between any two 0's of a 'lower' eigenfunction.

Consider a differential operator

$$Lf = \frac{d}{dx}\, p\, \frac{d}{dx} f$$

where p is a *positive* real-valued function on some large interval on the real line.

Theorem: Suppose that f, g are two twice-continuously-differentiable satisfy

$$Lf = \lambda \cdot f \qquad Lg = \mu \cdot g$$

with $\mu < \lambda \le 0$. Suppose that $f(a) = f(b) = 0$ for some $a < b$. Then the function g vanishes in the interior of the interval $[a, b]$.

Proof: Of course we assume that f is not identically 0. Let a, b be two consecutive 0's of f, so $f(a) = 0$, $f(b) = 0$, and $f(t) \neq 0$ for $a < t < b$. Without loss of generality, we may suppose that f is positive in the interval $[a, b]$, so $f'(a) \ge 0$ and $f'(b) \le 0$. For twice continuously differentiable functions f, g on $[a, b]$, integration by parts gives

$$\int_a^b Lf(t) \cdot g(t)\; dt = [p \cdot (f'g - fg')]_a^b + \int_a^b f(t) \cdot Lg(t)\, dt$$

This identity gives

$$\lambda \cdot \int_a^b |f(t)|^2\, dt = [pf'f]_a^b - \int_a^b p \cdot |f'(t)|^2\, dt < 0$$

since p is positive and f vanishes at the endpoints a, b. Thus, indeed, $\lambda < 0$. Suppose g does not vanish in the interior of the interval. Without loss of generality we may suppose that g also is positive at or just to the right of a. Then g is non-negative on the whole interval, and positive in the interior. Rearranging the identity above gives

$$(\lambda - \mu) \cdot \int_a^b f(t)\, g(t)\, dt = p(b)f'(b)g(b) - p(b)f(b)g'(b) - p(a)f'(a)g(a) + p(a)f(a)g'(a)$$

$$= p(b)f'(b)g(b) - p(a)f'(a)g(a) \le 0$$

since p is positive, g is non-negative, $f'(b) \le 0$, and $f'(a) \ge 0$. But, since f and g are positive in the interior of the interval, and since $\mu < \lambda$, the left-hand side is strictly positive. Contradiction. ♣