von Neumann density theorem

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**Theorem:** Let \( V \) be a Hilbert space, and \( A \) a \( \ast \)-stable C-algebra of continuous linear operators on \( V \) containing the constants. Then for any collection \( \{v_n\} \) of vectors in \( V \) with \( \sum_n |v_n|^2 < \infty \), for all \( \varepsilon > 0 \) and for all \( T \) in the double commutant \( A'' \) of \( A \) in \( \text{End}_c(V) \), there is \( x \in A \) such that

\[
\sum_n |(T - x)v_n|^2 < \varepsilon
\]

**Remark:** This asserts that \( A \) is dense in \( A'' \) in the ultra-strong topology on operators on \( V \), given by seminorms \( p_{\{v_n\}} \) for vectors \( v_n \) with \( \sum_n |v_n|^2 < \infty \), defined by

\[
p_{\{v_n\}}(T)^2 = \sum_n |Tv_n|^2
\]

Recall that the strong topology is given by the seminorms \( p_v \) for \( v \in V \) defined by

\[
p_v(T) = |Tv|
\]

V.S. Varadarajan has remarked that A. Weil first noted the appearance of the strong operator topology in such a circumstance. See J. Dixmier's *von Neumann algebras* for more information.

**Proof:** First, we claim that for each \( v \in V \)

\[
A''v \subset \overline{Av}
\]

where the overbar denotes closure in \( V \). The closure \( \overline{Av} \) is \( A \)-stable, and \( A \) is \( \ast \)-stable so the orthogonal complement to \( \overline{Av} \) is also \( A \)-stable. Thus, the orthogonal projection \( P : V \to \overline{Av} \) to it commutes with every element of \( A \): for \( v \in V \)

\[
PTv = PT(Pv + (1_V - P)v) = TPv + 0 = TPv
\]

since \( PV \) and \( (1_V - P)V \) are \( T \)-stable. Then \( T \in A'' \) commutes with \( P \). Thus, as \( 1_V \in A \),

\[
Tv = T \cdot 1_V v = T(Pv) = P(Tv) = P(Tv) \in \overline{Av}
\]

So there is a sequence \( a_1, a_2, \ldots \) in \( A \) such that \( a_nv \to Tv \). Thus \( A \) is dense in \( A'' \) in the strong operator topology. The argument can be enhanced to prove density of \( A \) in \( A'' \) in the ultra-strong topology, as follows.

Let \( W \) be the Hilbert space of sequences \( \{v_n : n \geq 1\} \) with \( \sum_n |v_n|^2 < \infty \). Let \( x \in A \) act diagonally on \( W \), by

\[
x^A \{v_n\} = \{xv_n\}
\]

Let \( p_n : W \to V \) be the orthogonal projection to the \( n \)-th component, and \( i_n : V \to W \) the imbedding of \( V \) at the \( n \)-th component. For \( S \in \text{End}_c(W) \) commuting with the diagonal action \( A^A \) of \( A \), for \( x \in A \)

\[
(p_n Sx^A)\{v_n\} = (p_m x^A S)\{v_n\} = (xp_m S)\{v_n\}
\]

And for \( v \in V \)

\[
(S_i n x)v = (Sx^A i_n)v = (x^A S i_n)v
\]

Thus, for all indices \( m, n \),

\[
p_m S_i n \in A'
\]

For \( T \in A'' \), we claim that the diagonal action

\[
T^A \{v_n\} = \{Tv_n\}
\]

lies in the double commutant \( (A^A)'' \). It suffices to prove that for \( S \) in the commutant \( (A^A)' \)

\[
p_m(ST^A - T^A S)i_n = 0 \in A'
\]

for all indices \( m, n \). Having already noted that \( p_m S_i n \) is in \( A' \),

\[
p_m ST^A i_m = (p_m S_i n)T = T(p_m S_i m) = p_n T^A S_i m
\]

as desired. Thus, by the first part of this proof, \( T^A \) can be approximated in the strong topology on \( W \) by elements of \( A^A \), so \( T \) can be approximated in the ultra-strong topology on \( V \) by elements of \( A \).  

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