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Weak C^k implies Strong C^{k-1}

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We show that weak C^k-ness implies (strong) C^{k-1}-ness for vector-valued functions with values in a *quasi-complete* locally convex topological vector space. In particular, weak smoothness implies smoothness. (Recall that a topological vector space is *quasi-complete* if every *bounded Cauchy net* is convergent.)

It seems that this theorem for the case of Banach-space-valued functions is well-known, at least folklorically, but the simple general case is at best apocryphal.

(If there were any doubt, the present sense of *weak differentiability* of a function f does not refer to distributional derivatives, but rather to differentiability of every scalar-valued function $\lambda \circ f$ where f is vector-valued and λ ranges over suitable continuous linear functionals.)

For clarity and emphasis, we recall some standard definitions. Let V be a topological vectorspace. A vector-valued function f on an open subset U of $\mathbf{R}$ is *differentiable* if, for each $x_o \in U$,

$$f'(x_o) = \lim_{x \to x_o} (x - x_o)^{-1} (f(x) - f(x_o))$$

exists. The function f is *continuously differentiable* if it is differentiable and if f' is continuous. A k-times continuously differentiable function is said to be C^k, and a continuous function is said to be C^o. A V-valued function is **weakly** C^k if for every $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ is C^k. Generally, as usual, for an $\mathbf{R}$-valued function f on an open subset U of $\mathbf{R}^n$ with $n \geq 1$, say that f is *differentiable* at $x_o \in U$ if there is a vector D_o in $\mathbf{R}^n$ so that (using the little-oh notation)

$$f(x) = f(x_o) + \langle x - x_o, D_o \rangle + o(x - x_o)$$

as $x \to x_o$, where $\langle , \rangle$ is the usual pairing $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$. Such f is *continuously* differentiable if the function $x_o \to D_o$ is a continuous $\mathbf{R}^n$-valued function.

Theorem: Let V be a quasi-complete locally convex topological vector space. Let f be a V-valued function defined on an interval $[a, c]$. Suppose that f is weakly C^k. Then the V-valued function f is (strongly) C^{k-1}.

First we need

Lemma: Let V be a quasi-complete locally convex topological vector space. Fix real numbers $a \leq b \leq c$. Let f be a V-valued function defined on $[a, b) \cup (b, c]$. Suppose that for each $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ has an extension to a function F_λ on the whole interval $[a, c]$ which is C^1. Then $f(b)$ can be chosen so that the extended $f(x)$ is (strongly) continuous on $[a, c]$.

Proof: For each $\lambda \in V^*$, let F_λ be the extension of $\lambda \circ f$ to a C^1 function on $[a, c]$. For each λ, the differentiability of F_λ implies that

$$\Phi_\lambda(x, y) = \frac{F_\lambda(x) - F_\lambda(y)}{x - y}$$

has a continuous extension $\tilde{\Phi}_\lambda$ to the compact set $[a, c] \times [a, c]$. Thus, the image C_λ of $[a, c] \times [a, c]$ under this continuous map is a compact subset of $\mathbf{R}$, so bounded. Thus, the subset

$$\{ \frac{\lambda f(x) - \lambda f(y)}{x - y} : x \neq y \} \subset C_\lambda$$

is also bounded in $\mathbf{R}$. Therefore, the set

$$E = \{ \frac{f(x) - f(y)}{x - y} : x \neq y \} \subset V$$

is weakly bounded. It is a standard fact (from Banach-Steinhaus, Hahn-Banach, and Baire category arguments) that weak boundedness implies (strong) boundedness in a locally convex topological vectorspace, so E is (strongly) bounded. Thus, for a (strong, balanced, convex) neighborhood N of 0 in V, there is t_o so that $(f(x) - f(y))/(x - y) \in tN$ for any $x \neq y$ in $[a, c]$ and any $t \geq t_o$. That is,

$$f(x) - f(y) \in (x - y)tN$$

Thus, given N and the t_o determined as just above, for $|x - y| < \frac{1}{t_o}$ we have

$$f(x) - f(y) \in N$$

That is, as $x \to 0$ the collection $f(x)$ is a bounded Cauchy net. Thus, by the quasi-completeness, we can define $f(b) \in V$ as the limit of the values $f(x)$. And in fact we see that for $x \to y$ the values $f(x)$ approach $f(y)$, so this extended version of f is continuous on $[a, c]$. ♣

Proof: (of theorem) Fix $b \in (a, c)$, and consider the function

$$g(x) = \frac{f(x) - f(b)}{x - b}$$

for $x \neq b$. The assumed weak C^2-ness implies that every $\lambda \circ g$ extends to a C^1 function on $[a, c]$. Thus, by the lemma, g itself has a continuous extension to $[a, c]$. In particular,

$$\lim_{x \to b} g(x)$$

exists, which exactly implies that f is differentiable at b. Thus, f is differentiable throughout $[a, c]$.

To prove the continuity of f', consider again the function of two variables (initially for $x \neq y$)

$$g(x, y) = \frac{f(x) - f(y)}{x - y}$$

The weak C^2-ness of f assures that g extends to a weakly C^1 function on $[a, c] \times [a, c]$. In particular, the function $x \to g(x, x)$ of (the extended) g is weakly C^1. This function is $f'(x)$. Thus, f' is weakly C^1, so is (strongly) C^o.

Suppose that we already know that f is C^ℓ, for $\ell < k - 1$. Then consider the ℓ^{th} derivative $g = f^{(\ell)}$ of f. This function g is at least weakly C^2, so is (strongly) C^1 by the first part of the argument. That is, f is at least $C^{\ell+1}$. ♣
