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**Weak $C^k$ implies Strong $C^{k-1}$**

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We show that weak $C^k$-ness implies (strong) $C^{k-1}$-ness for vector-valued functions with values in a quasi-complete locally convex topological vector space. In particular, weak smoothness implies smoothness. (Recall that a topological vector space is quasi-complete if every bounded Cauchy net is convergent.)

It seems that this theorem for the case of Banach-space-valued functions is well-known, at least folklorically, but the simple general case is at best apocryphal.

(If there were any doubt, the present sense of weak differentiability of a function $f$ does not refer to distributional derivatives, but rather to differentiability of every scalar-valued function $\lambda \circ f$ where $f$ is vector-valued and $\lambda$ ranges over suitable continuous linear functionals.)

For clarity and emphasis, we recall some standard definitions. Let $V$ be a topological vector space. A vector-valued function $f$ on an open subset $U$ of $\mathbb{R}$ is differentiable if, for each $x_o \in U$,

$$f'(x_o) = \lim_{x \to x_o} \frac{(f(x) - f(x_o))}{x - x_o}$$

exists. The function $f$ is continuously differentiable if it is differentiable and if $f'$ is continuous. A $k$-times continuously differentiable function is said to be $C^k$, and a continuous function is said to be $C^0$. A $V$-valued function is weakly $C^k$ if for every $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ is $C^k$. Generally, as usual, for an $\mathbb{R}$-valued function $f$ on an open subset $U$ of $\mathbb{R}^n$ with $n \geq 1$, say that $f$ is differentiable at $x_o \in U$ if there is a vector $D_o$ in $\mathbb{R}^n$ so that (using the little-o notation)

$$f(x) = f(x_o) + \langle x - x_o, D_o \rangle + o(x - x_o)$$

as $x \to x_o$, where $\langle \cdot, \cdot \rangle$ is the usual pairing $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Such $f$ is continuously differentiable if the function $x_o \to D_o$ is a continuous $\mathbb{R}^n$-valued function.

**Theorem:** Let $V$ be a quasi-complete locally convex topological vector space. Let $f$ be a $V$-valued function defined on an interval $[a, c]$. Suppose that $f$ is weakly $C^k$. Then the $V$-valued function $f$ is (strongly) $C^{k-1}$.

First we need

**Lemma:** Let $V$ be a quasi-complete locally convex topological vector space. Fix real numbers $a \leq b \leq c$. Let $f$ be a $V$-valued function defined on $[a, b] \cup (b, c]$. Suppose that for each $\lambda \in V^*$ the scalar-valued function $\lambda \circ f$ has an extension to a function $F_\lambda$ on the whole interval $[a, c]$ which is $C^1$. Then $f(b)$ can be chosen so that the extended $f(x)$ is (strongly) continuous on $[a, c]$.

**Proof:** For each $\lambda \in V^*$, let $F_\lambda$ be the extension of $\lambda \circ f$ to a $C^1$ function on $[a, c]$. For each $\lambda$, the differentiability of $F_\lambda$ implies that

$$\Phi_\lambda(x, y) = \frac{F_\lambda(x) - F_\lambda(y)}{x - y}$$

has a continuous extension $\Phi_\lambda$ to the compact set $[a, c] \times [a, c]$. Thus, the image $C_\lambda$ of $[a, c] \times [a, c]$ under this continuous map is a compact subset of $\mathbb{R}$, so bounded. Thus, the subset

$$\left\{ \frac{\lambda f(x) - \lambda f(y)}{x - y} : x \neq y \right\} \subset C_\lambda$$

is also bounded in $\mathbb{R}$. Therefore, the set

$$E = \left\{ \frac{f(x) - f(y)}{x - y} : x \neq y \right\} \subset V$$

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is weakly bounded. It is a standard fact (from Banach-Steinhaus, Hahn-Banach, and Baire category arguments) that weak boundedness implies (strong) boundedness in a locally convex topological vector space, so $E$ is (strongly) bounded. Thus, for a (strong, balanced, convex) neighborhood $N$ of 0 in $V$, there is $t_o$ so that $(f(x) - f(y))/(x - y) \in tN$ for any $x \neq y$ in $[a, c]$ and any $t \geq t_o$. That is,

$$f(x) - f(y) \in (x - y)tN$$

Thus, given $N$ and the $t_o$ determined as just above, for $|x - y| < \frac{1}{t_o}$ we have

$$f(x) - f(y) \in N$$

That is, as $x \to 0$ the collection $f(x)$ is a bounded Cauchy net. Thus, by the quasi-completeness, we can define $f(b) \in V$ as the limit of the values $f(x)$. And in fact we see that for $x \to y$ the values $f(x)$ approach $f(y)$, so this extended version of $f$ is continuous on $[a, c]$. ♦

Proof: (of theorem) Fix $b \in (a, c)$, and consider the function

$$g(x) = \frac{f(x) - f(b)}{x - b}$$

for $x \neq b$. The assumed weak $C^2$-ness implies that every $\lambda \circ g$ extends to a $C^1$ function on $[a, c]$. Thus, by the lemma, $g$ itself has a continuous extension to $[a, c]$. In particular,

$$\lim_{x \to b} g(x)$$

exists, which exactly implies that $f$ is differentiable at $b$. Thus, $f$ is differentiable throughout $[a, c]$.

To prove the continuity of $f'$, consider again the function of two variables (initially for $x \neq y$)

$$g(x, y) = \frac{f(x) - f(y)}{x - y}$$

The weak $C^2$-ness of $f$ assures that $g$ extends to a weakly $C^1$ function on $[a, c] \times [a, c]$. In particular, the function $x \to g(x, x)$ of (the extended) $g$ is weakly $C^1$. This function is $f'(x)$. Thus, $f'$ is weakly $C^1$, so is (strongly) $C^\alpha$.

Suppose that we already know that $f$ is $C^\ell$, for $\ell < k - 1$. Then consider the $\ell$th derivative $g = f^{(\ell)}$ of $f$. This function $g$ is at least weakly $C^2$, so is (strongly) $C^1$ by the first part of the argument. That is, $f$ is at least $C^{\ell+1}$. ♦