Weil-Schwartz envelopes for rapidly decreasing functions
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**Definition:** A complex-valued function \( f \) on \( \mathbb{R}^n \) is **rapidly decreasing** if for every \( m \geq 0 \)
\[
\sup_{x \in \mathbb{R}^n} |x|^m |f(x)| < \infty
\]
where the first \(|*|\) is the usual norm on \( \mathbb{R}^n \) and the second is the absolute value on \( \mathbb{C} \).

**Theorem:**
- Given a rapidly-decreasing function \( f \) on \( \mathbb{R}^n \) there is a Schwartz function \( \varphi \) such that \(|\varphi| \geq |f|\).
- Given a countable collection \( \{f_i\} \) of rapidly decreasing functions, there is a positive monotone decreasing Schwartz function \( \varphi \) such that \( f_i/\varphi \) is rapidly decreasing for every index \( i \).
- Every Schwartz function on \( \mathbb{R}^n \) is a product of two Schwartz functions.

**Proof:** Let
\[
c_m = \sup_x (|x|^2 + 1)^m \cdot |f(x)| < \infty
\]
For \( 0 \leq r \in \mathbb{R} \) define
\[
F(r) = \inf_m \frac{c_m}{(1 + r)^m}
\]
Since for every \( m \)
\[
|f(x)| \leq \frac{c_m}{(1 + |x|^2)^m}
\]
we have
\[
|f(x)| \leq F(|x|^2)
\]
For any collection of numbers \( c_m \), and infimum such as \( F \) is rapidly decreasing, since the exponents in the denominators are unbounded. Since each \( c_m/(1 + r)^m \) is monotone decreasing as \( r \to \infty \), so is \( F \). Since \( F \)
is the infimum of a countable collection of continuous functions, it is measurable.

Let \( \beta \) be a smooth non-negative function supported on \([-1, +1]\), with total mass 1. Take
\[
\varphi(r) = (F * \beta)(r) = \int_{\mathbb{R}} F(r - t) \beta(t) \, dt
\]
Certainly \( \varphi \) is positive. The monotonicity of \( F \) implies that
\[
F(r) \leq \varphi(r - 1) \quad (\text{for } r \geq 1)
\]
We claim that \( \varphi \) is a Schwartz function on \( \mathbb{R} \). It has derivatives
\[
\varphi^{(n)}(r) = (F * \beta^{(n)})(r)
\]
which are smooth (since \( \beta \) is) and rapidly decreasing (since \( F \) is). Let \( \varphi_2 \) be any non-negative Schwartz function on \( \mathbb{R}^n \) such that for \(|x| \leq 1\) we have \( \varphi_2(x) \geq |f(x)| \). Then
\[
|f(x)| \leq F(|x|^2) \leq \varphi(|x|^2 - 1) + \varphi_2(|x|^2)
\]
Let
\[
\Phi(x) = \varphi(|x|^2 - 1) + \varphi_2(|x|^2)
\]
Then \( \Phi \) is a Schwartz function and
\[
|\Phi(y)| \leq |\Phi(x)| \quad (\text{for } 1 \leq |x| \leq |y|)
\]
To ensure that $\Phi$ is positive and monotone, we may add to it a sufficiently large positive multiple of $e^{-|x|^2}$ without harming any of the other properties. This proves the first assertion.

Given a countable collection of rapidly decreasing functions $f_i$, we may suppose without loss of generality (by the first assertion of this theorem) that $f_i$ is a positive monotone Schwartz function. By replacing $f_i$ by $\sup_{1 \leq j \leq i} f_j$ we may suppose that $f_i \leq f_j$ for $i < j$. Inductively define $M_1 = 1$ and

$$M_n = \text{smallest } t \geq 1 + M_{n-1} \text{ such that } |x|^n |f(x)| \leq 1 \text{ for } |x| \geq M_n$$

Define

$$f(x) = f_n(x) \text{ (for } M_n \leq |x| < M_{n+1})$$

Claim that $f$ is rapidly decreasing. Indeed, given a positive integer $m$, for any $n$, for $M_n \leq |x| < M_{n+1}$,

$$|x|^m f(x) = |x|^m f_n(x) \leq |x|^n f_n(x) \leq 1$$

Again using the first part of the theorem, there is a positive monotone Schwartz function $\varphi$ such that $\varphi \geq |f|$. Then for each index $i$ the ratio $f_i/\varphi$ is clearly bounded.

The square root $\varphi_2(x) = \sqrt{\varphi(x)}$ is still monotone, rapidly decreasing, and continuous. Since $f_i/\varphi$ is bounded,

$$f_i/\varphi_2 = \varphi_2 \cdot f_i/\varphi$$

shows that each $f_i/\varphi_2$ is rapidly decreasing. Invoking the first assertion of the theorem yet again, there is a positive monotone Schwartz function $\psi$ such that $\psi \geq |\varphi_2|$. This yields the second assertion.

Given a Schwartz function $f$, consider the countable family of rapidly decreasing functions

$$\Phi = \{|\varphi(x)|^{1/m} : m = 1, 2, 3, \ldots, \varphi \text{ a derivative of } f\}$$

By the second assertion of the theorem, there is a positive monotone Schwartz function $\psi$ such that $\varphi/\psi$ is rapidly decreasing for every $\varphi \in \Phi$. To verify that $F = f/\psi$ is a Schwartz function requires the rapid decrease of all derivatives of $F$. Given a monomial $D$ in the partial derivative operators $\partial/\partial x_i$, by induction on the order $n$ of $D$ one can show that $DF$ is of the form

$$DF = \left(\sum_{\alpha} P_\alpha(D^{\alpha} f)\right)/\psi^{n+1}$$

where $P_\alpha$ is a polynomial in $\psi$ and its derivatives, $D^{\alpha}$ is a monomial in the operators $D_i$, and the sum is finite. Since each $P_\alpha$ is bounded, it would suffice to show that any ration of the form $Df/\psi^n$ is of rapid decrease. By the defining property of $\psi$ the function $|Df|^{1/n}/\psi$ is of rapid decrease, so its $n^{th}$ power is, as well. That is, $f/\psi$ is Schwartz, and $f = \psi \cdot f/\psi$.

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