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00 exercises, homological ...

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[00.1] Show that the usual elementary definitions of polynomial rings $R[x]$ do satisfy the universal property characterizing a free R-algebra on x, namely, that for *any* R-algebra A and chosen element $a_o \in A$, there is a unique R-algebra homomorphism $f : R[x] \to A$ such that $fx = a_o$.

[00.2] A **derivation** $D : k[x] \to k[x]$ on the polynomial ring $k[x]$ over a field k is meant to resemble a *derivative*, namely, it is a k-linear map, it annihilates the *constants* k, and satisfies Leibniz' rule $D(fg) = Df \cdot g + f \cdot Dg$. Prove that Dx determines D on $k[x]$, and classify the possible values of Dx.

[00.3] Prove that a polynomial P in one variable with coefficients in a field k has a repeated root if and only if P and DP have a common factor, where $D : k[x] \to k[x]$ is a derivation such that $Dx = 1$.

[00.4] Show that the set-theory definition $(a, b) = \{\{a\}, \{a, b\}\}$ of *ordered pair* (a, b) really does have the property we want, namely, that $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$. Note that the case $a = b$ entails $\{a, b\} = \{a\}$, and $\{\{a\}, \{a, b\}\} = \{\{a\}, \{a\}\} = \{a\}$.

[00.5] Show that the first uncountable ordinal ω_1 is *regular*, in the sense that it is *not* a countable union/limit of smaller (therefore, countable) ordinals.

[00.6] Show that $\mathbb{Q}$ (with addition) is a torsion-free, (but) non-free, not-finitely-generated $\mathbb{Z}$-module.

[00.7] Let k be a field and W a k-vector-subspace of a k-vectorpace V. Show that any k-basis of W extends to a k-basis of V. Treat infinite-dimensional vector spaces by some equivalent of transfinite induction.

[00.8] Let $0 \longrightarrow V_n \longrightarrow V_{n-1} \longrightarrow \ldots \longrightarrow V_1 \longrightarrow V_0 \longrightarrow 0$ be an *exact sequence* of vectorspaces over a field k. Show that

$$\dim_k V_n - \dim_k V_{n-1} + \dim_k V_{n-2} - \ldots + (-1)^{n-1} \dim_k V_1 + (-1)^n \dim V_0 \;=\; 0$$

[00.9] For a field k, determine a short-as-possible free resolution of the *trivial* $k[x, y]$-module k, on which x and y both act by 0, where $k[x, y]$ is the polynomial ring in two variables.

[00.10] Show that a map $i : A \longrightarrow B$ of sets is *injective* if and only if, for every pair of maps f, g of any other set X to A, $i \circ f = i \circ g$ if and only if $f = g$. With any sort of objects, not only sets, the latter property is that i is a **monomorphism**.

[00.11] Show that a map $q : A \longrightarrow B$ of sets is *surjective* if and only if, for every pair of maps f, g of B to any other set X, $f \circ q = g \circ q$ if and only if $f = g$. With any sort of objects, not only sets, the latter property is that q is an **epimorphism**.

[00.12] Do the previous two exercises for other types of objects, for example, abelian groups, not-necessarily-abelian groups.
