(*September 23, 2009*)

01 Examples/exercises

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[01.1] Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence of *free* $\mathbb{Z}$-modules. Show that, for any positive integer n,

$$0 \longrightarrow \frac{A}{nA} \longrightarrow \frac{B}{nB} \longrightarrow \frac{C}{nC} \longrightarrow 0$$

is still *exact*.

[01.2] Let I be an ideal in a commutative ring R. For an R-module M, let M^I be the submodule on which I acts by 0, and let M_I be the largest quotient on which I acts by 0, namely $M/I \cdot M$. Prove the *adjunction relation*

$$\mathrm{Hom}_R(M_I, N) \approx \mathrm{Hom}_(M, N^I) \qquad \text{(for } R\text{-modules } M, N)$$

[01.3] Let $k \subset K$ be fields. Let R be a right adjoint to the forgetful functor from K-vectorspaces to k-vectorspaces. Prove that

$$\dim_K RU = \dim_k U \qquad \text{(for all } k\text{-vectorspaces } U)$$

In fact, the same argument should work for free modules over commutative rings.

[01.4] Show that $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}_n \approx \mathbb{Z}/\gcd(m,n)$.

[01.5] Let M and N be two free modules on generating sets S and T, over a commutative ring R. Show that $M \otimes_R N$ is free with generating set $S \times T$.

[01.6] Show that the product of sets or groups is the usual cartesian product.

[01.7] Show that the coproduct of sets is the *disjoint union*.

[01.8] Show that the coproduct of abelian groups is the *direct sum*.

[01.9] Show that the coproduct of not-necessarily abelian groups is the *free product*.

[01.10] Prove that there is a natural isomorphism $A \otimes (B \otimes C) \approx (A \otimes B) \otimes C$ with $\mathbb{Z}$-modules A, B, C.

[01.11] Prove the Eilenberg-MacLane adjunction in the case of $\mathbb{Z}$-modules

$$\mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(A \otimes_{\mathbb{Z}} B, C) \approx_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(A, \mathrm{Hom}_{\mathbb{Z}-\mathrm{mod}}(B, C))$$

[01.12] Prove that the universal algebra UV of a vector space V over a field k *exists* by showing that the *construction* of UV as $\bigoplus_n \bigotimes^n V$ succeeds.
