1. Simultaneous eigenvectors for finite abelian groups

For a single linear operator $T$ on a complex vector space $V$, and for a complex number $\lambda$, the $\lambda$-eigenspace $V_\lambda$ of $T$ on $V$ is

$$V_\lambda = \{ v \in V : T v = \lambda \cdot v \}$$

[1.1] Unitarizability, diagonalizability of finite-order operators
Various natural hypotheses assure that $V$ is a direct sum of $T$-eigenspaces, that is, that $T$ is diagonalizable, as opposed to having any Jordan blocks. Here, a natural hypothesis is that $T$ is of finite order, that is, $T^n = 1$ for some $1 \leq n \in \mathbb{Z}$. With this hypothesis, we can construct an complex-hermitian inner product $\langle \cdot, \cdot \rangle$ on $V$ such that $T$ is unitary in the sense that

$$\langle T v, T w \rangle = \langle v, w \rangle \quad \text{(for all } v, w \in V \text{)}$$

Take an arbitrary hermitian inner-product $\langle \cdot, \cdot \rangle_o$, and create a $T$-invariant one by averaging:

$$\langle v, w \rangle = \sum_{\ell=1}^{n} \langle T^\ell v, T^\ell w \rangle_o$$

The averaging process does not disturb positive-definiteness or hermitian-ness. The $T$-invariance, that is, unitariness of $T$, is easy, using $T^n = 1$,

$$\langle T v, T w \rangle = \sum_{\ell=1}^{n} \langle T^\ell v, T^\ell T w \rangle_o = \sum_{\ell=2}^{n+1} \langle T^\ell v, T^\ell w \rangle_o = \sum_{\ell=2}^{n} \langle T^\ell v, T^\ell w \rangle_o + \langle T^{n+1} v, T^{n+1} w \rangle_o$$

$$= \sum_{\ell=2}^{n} \langle T^\ell v, T^\ell w \rangle_o + \langle T v, T w \rangle_o = \sum_{\ell=1}^{n} \langle T^\ell v, T^\ell w \rangle_o = \langle v, w \rangle$$

Then the spectral theorem for unitary operators implies that $T$ is diagonalizable, that is, that $V$ is an orthogonal direct sum of $T$-eigenspaces:

$$V = \bigoplus_\lambda V_\lambda \quad \text{(with an orthogonal direct sum)}$$

[1.2] Commuting operators
Another linear operator $S$ commuting with $T$ stabilizes each $T$-eigenspace $V_\lambda$: for $v \in V_\lambda$:

$$T(Sv) = (TS)v = (ST)v = S(Tv) = S(\lambda v) = \lambda \cdot S v$$

since the linearity of $S$ is that $S$ commutes with scalar multiplication.
[1.3] Finite abelian groups of operators  We want to prove that a finite abelian group $G$ of operators on a finite-dimensional complex vectorspace $V$ is simultaneously diagonalizable. That is, we claim that $V$ is a direct sum of simultaneous eigenspaces for all operators in $G$.

In this situation, the notion of eigenvalue must be a little more complicated than individual numbers: for each $g \in G$, there must be a number $\lambda_g \in \mathbb{C}$. That is, an eigenvalue is really a map $g \mapsto \lambda_g$ from $G$ to $\mathbb{C}$. In this context, for two eigenvalues $\lambda, \mu$ to be distinct means that $\lambda_g \neq \mu_g$ for some $g \in G$ (not necessarily for all $g \in G$).

Further, if there is a non-zero eigenvector $v$ for a given collection of eigenvalues $g \mapsto \lambda_g$, then $g \mapsto \lambda_g$ is a group homomorphism from $G$ to $\mathbb{C} \times$: $\lambda_{gh} \cdot v = (gh) \cdot v = g(\lambda_h \cdot v) = \lambda_h \cdot gv = \lambda_h \lambda_g \cdot v = \lambda_g \lambda_h \cdot v$

That is, $g \mapsto \lambda_g$ is a character of $G$. Let $\hat{G}$ denote the collection of all characters of $G$: 

$$\hat{G} = \\{ \text{group homomorphisms } \chi : G \rightarrow \mathbb{C}^\times \}$$

and for a character $\chi$ let the $\chi$-eigenspace in $V$ be

$$V_\chi = \{ v \in V : gv = \chi(g) \cdot v, \text{ for all } g \in G \}$$

Thus, our claim is that for a finite abelian group $G$ of linear operators on finite-dimensional complex vector space $V$,

$$V = \bigoplus_{\chi \in \hat{G}} V_\chi$$

As with a single operator generating a finite group of operators, we can make a hermitian inner product on $V$ by averaging over $G$ a given, arbitrary hermitian inner product $\langle \cdot, \cdot \rangle_o$: put

$$\langle v, w \rangle = \sum_{g \in G} \langle gv, gw \rangle \quad ( \text{for } v, w \in V)$$

The $G$-invariance follows by changing variables in the summation: for $h \in G$,

$$\langle hv, hw \rangle = \sum_{g \in G} \langle ghv, ghw \rangle = \sum_{g \in G} \langle gv, gw \rangle \quad (\text{replacing } g \text{ by } gh^{-1})$$

Thus, all the operators in $G$ are unitary with respect to $\langle \cdot, \cdot \rangle$.

The proof: suppose we have the direct sum decomposition for vector spaces of dimension $< n$. Let $V$ be of dimension $n$. First, a silly case: if all operators $g \in G$ are scalar, then every vector is a simultaneous eigenvector for all the operators in $G$, and we are done. So now consider the (serious) case that not all operators in $G$ are scalar. Let $g \in G$ be a non-scalar operator. By the spectral theorem for unitary operators, $V$ has an orthogonal decomposition into eigenspaces for $g$, implicitly with different eigenvalues. Since $g$ is non-scalar, every one of these eigenspaces has dimension $< n$. By induction, and by the fact that the operators all commute, each such eigenspace decomposes as an orthogonal direct sum of simultaneous eigenspaces for $G$. Thus, the whole space $V$ is an orthogonal direct sum of simultaneous eigenspaces. This completes the induction.
2. Cancellation lemma, orthogonality of distinct characters

Let $G$ be a finite group, not necessarily abelian. The cancellation lemma is

[2.1] Lemma: For a non-trivial group homomorphism $\sigma : G \to \mathbb{C}^\times$,

$$\sum_{g \in G} \sigma(g) = 0$$

Proof: Since $\sigma$ is not identically 1, there is $g_o \in G$ such that $\sigma(g_o) \neq 1$. Then

$$\sum_{g \in G} \sigma(g) = \sum_{g \in G} \sigma(g_o g) = \sum_{g \in G} \sigma(g_o) \sigma(g) = \sigma(g_o) \sum_{g \in G} \sigma(g)$$

by replacing $g$ by $g_o g$ in the sum, using the fact that left multiplication by $g_o$ is a bijection of $G$ to itself. Subtracting,

$$(1 - \sigma(g_o)) \cdot \sum_{g \in G} \sigma(g) = 0$$

Since $\sigma(g_o) \neq 1$, necessarily the sum is 0. ///

[2.2] Corollary: Let $\sigma \neq \tau$ be group homomorphisms $G \to \mathbb{C}^\times$. Then

$$\sum_{g \in G} \sigma(g) \tau(g) = 0$$

Proof: Since $G$ is finite, there is $N$ such that $g^N = e$ for every $g \in G$. Thus,

$$\tau(g)^N = \tau(g^N) = \tau(e) = 1$$

Thus, $\tau(g)$ is a root of unity, and $|\tau(g)| = 1$. In particular, $\tau(g) = \tau(g)^{-1}$. Then $\sigma\tau = \sigma\tau^{-1}$ is a character of $G$, and is not the trivial character. The cancellation lemma gives the vanishing. ///

[2.3] Remark: Despite the simplicity of the arguments above, the cancellation and orthogonality devices are remarkably useful in applications.

3. Representations of finite abelian groups

It is useful to consider a slight shift of viewpoint. Instead of having a finite abelian group $G$ of linear automorphisms of a fixed complex vector space $V$, we might fix a finite abelian group $G$, and consider various group homomorphisms $\rho : G \to \text{Aut}_C V$ for various complex vector spaces $V$. The pair $\rho, V$ is a representation of $G$.

Although it would be most strictly correct to write the action of $g \in G$ on $v \in V$ via $\rho$ as $\rho(g)(v)$, context should be sufficient to allow writing simply $g \cdot v$ or $gv$.

Similarly, it is common to write $V_\rho$ for $V$ to emphasize the role played by $\rho$, or to use other notational devices, but, in fact, context should make dependencies clear.

The collection $\hat{G}$ of characters $G \to \mathbb{C}^\times$ makes sense for arbitrary groups $G$, with or without realizing them as subgroups of linear automorphisms of vector spaces.
For finite abelian $G$, the image $\rho(G)$ of $G$ in the group of automorphisms of $V$ is still finite abelian, and, as a corollary of the result for finite abelian subgroups of $\text{Aut}_C V$,

$$V = \bigoplus_{\chi \in \hat{G}} V_\chi$$

where

$$V_\chi = \{v \in V : g \cdot v = \chi(g) \cdot v, \text{ for all } g \in G\}$$

### 4. Fourier expansions on finite abelian groups

Let $G$ be a finite abelian group, and $L^2(G)$ the complex vectorspace of complex-valued functions$^1$ on $G$, with inner product$^2$

$$\langle f, \varphi \rangle = \sum_{x \in G} f(x) \overline{\varphi(x)}$$

Again, a character $\chi$ on a group is a group homomorphism$^3$

$$\chi : G \rightarrow \mathbb{C}^\times$$

and $\hat{G}$ is the collection of characters $\chi : G \rightarrow \mathbb{C}^\times$. For $f$ a complex-valued function on $G$, the Fourier transform $\hat{f}$ of $f$ is the function on $\hat{G}$ defined by

$$\hat{f}(\chi) = \langle f, \chi \rangle \quad \text{(for } \chi \in \hat{G})$$

The Fourier expansion or Fourier series of $f$ is

$$f \sim \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi$$

**[4.1] Theorem:** On a finite abelian group, the Fourier expansion of a complex-valued function $f$ represents $\hat{f}$, in the sense that, for every $g \in G$,

$$f(g) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi(g)$$

The elements of $\hat{G}$ form an orthogonal basis for $L^2(G)$. In particular, the Fourier coefficients are unique.

$^1$ In general, for a space $X$ with some sort of integral on it, the notation $L^2(X)$ means functions $f$ so that $\int_X |f|^2 < \infty$. On finite sets integrals become sums, possibly weighted, and this finiteness condition becomes vacuous. Nevertheless, it is good to use this notation as a reminder of the larger context.

$^2$ The notation $L^2(G)$ is meant to suggest the presence of the inner product on this space of functions. On a general space $X$ with an integral, the inner product is $\langle f_1, f_2 \rangle = \int_X f_1 \overline{f_2}$.

$^3$ The term character has different meanings in different contexts. The simplest sense is a group homomorphism to $\mathbb{C}^\times$. However, an equally important use is for the trace of a group homomorphism $\rho : G \rightarrow GL_n(k)$ from $G$ to invertible $n$-by-$n$ matrices with entries in a field $k$. In the latter sense,

$$(\text{character of } \rho)(g) = \text{trace} \left( \rho(g) \right)$$

For infinite-dimensional representations, further complications appear. Context is always necessary to know which sense is intended.
abelian translation action loses some of its significance, since for the information that the two together could give. Also, for abelian groups, the inverse in the definition of the left translation is for

to reduce clutter, the action of $g \in G$ on functions $f$ may be written simply $gf$ or $g \cdot f$. The associativity property

$$((gh)f)(x) = f((xg)h) = f((xg)h) = (hf)(xg) = (g(hf))(x)$$

The associativity property is equivalent to the assertion that the map $g \to T_g$ is a group homomorphism from $G$ to $\mathbb{C}$-linear automorphisms of $L^2(G)$ (and that the identity element of $g$ acts trivially).

Since $g \to T_g$ is a group homomorphism, the abelian-ness of $G$ implies that the linear maps $T_g$, $T_h$ commute: since $gh = hg$,

$$T_g \circ T_h = T_{gh} = T_{hg} = T_h \circ T_g \quad \text{(for all } g, h \in G)$$

Since $G$ is finite, there is a positive integer $N$ such that, for all $g \in G$, $g^N = e \in G$. Thus,

$$\chi(g)^N = \chi(g^N) = \chi(e) = 1 \quad \text{(for any } \chi \in \hat{G})$$

[4] For non-abelian groups $G$, there are two translation actions, namely, left and right

$$\begin{cases}
T_g^{\text{right}} f(x) = f(xg) \\
T_g^{\text{left}} f(x) = f(g^{-1} x)
\end{cases}$$

The inverse in the left translation is for associativity

$$T_{gh}^{\text{left}} f = T_g \left(T_h f\right)$$

For abelian groups, the two translation actions become essentially the same thing, insofar as either alone gives all the information that the two together could give. Also, for abelian groups, the inverse in the definition of the left translation action loses some of its significance, since for abelian groups $g \to g^{-1}$ is a group automorphism.
That is, the values of $\chi$ lie on the unit circle in $\mathbb{C}^\times$, so $|\chi(g)| = 1$. In particular, $\chi$ is unitary in the sense that

$$\chi(g)^{-1} = \overline{\chi(g)}$$

We claim that the linear operators $T_g$ are also unitary, in the sense that

$$\langle T_g f, T_g F \rangle = \langle f, F \rangle \quad \text{for } g \in G, \text{ functions } f, F$$

To prove this, compute directly:

$$\langle T_g f, T_g F \rangle = \sum_{h \in G} (T_g f)(h) \overline{(T_g F)(h)} = \sum_{h \in G} f(hg) \overline{F(hg)}$$

Change variables in the sum, by replacing $h$ by $hg^{-1}$. Here the fact that $G$ is a group is used: $g^{-1}$ exists, and is closed under the group law:

$$\sum_{h \in G} f(hg) \overline{F(hg)} = \sum_{h \in G} f(h) \overline{F(h)} = \langle f, F \rangle$$

proving the unitarity.

From above, in the case that $V = L^2(G)$,

$$L^2(G) = \bigoplus_{\chi \in \hat{G}} L^2(G)_\chi$$

We will show that each $L^2(G)_\chi$ is one-dimensional, spanned by $\chi$ itself.

On one hand, every $\chi \in \hat{G}$ is a complex-valued function on $G$, so is in $L^2(G)$. Indeed, $\chi \in L^2(G)_\chi$:

$$(T_g \chi)(h) = \chi(hg) = \chi(h) \chi(g) = \chi(g) \chi(h) \quad \text{(since } \mathbb{C}^\times \text{ is abelian)}$$

On the other hand, $L^2(G)_\chi$ is exactly scalar multiples $\mathbb{C} \cdot \chi$ of $\chi$: for $f \in L^2(G)_\chi$,

$$f(g) = f(e \cdot g) = (T_g f)(e) = \chi(g) \cdot f(e) = f(e) \cdot \chi(g) \quad \text{(identity } e \in G)$$

That is,

$$f = f(e) \cdot \chi \quad \text{(for } f \in L^2(G)_\chi)$$

By the orthogonality of $V_\chi$ and $V_\tau$ for distinct $\chi, \tau$, the characters are an orthogonal basis for $L^2(G)$. Their lengths are readily determined, using the earlier-noted unitariness $\overline{\chi} = \chi^{-1}$:

$$\langle \chi, \chi \rangle = \sum_{g \in G} \chi(g) \cdot \overline{\chi(g)} = \sum_{g \in G} \chi(g) \cdot \chi(g)^{-1} = \sum_{g \in G} 1 = |G|$$

Any $f \in L^2(G)$ is a linear combination of orthogonal basis elements $e_i$

$$f = \sum_i \frac{\langle f, e_i \rangle \cdot e_i}{\langle e_i, e_i \rangle}$$

Using the orthogonal basis $\chi \in \hat{G}$,

$$f = \sum_{\chi \in \hat{G}} \frac{\langle f, \chi \rangle \cdot \chi}{\langle \chi, \chi \rangle} = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \langle f, \chi \rangle \cdot \chi$$
This is an equality of functions on the finite set \( G \), and \( \hat{f}(\chi) \) is defined to be \( \langle f, \chi \rangle \), so
\[
f(g) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \langle f, \chi \rangle \cdot \chi(g) = \frac{1}{|G|} \sum_{\chi \in \hat{G}} \hat{f}(\chi) \cdot \chi(g) \quad \text{(for all } g \in G)\]
This proves the representability of functions on finite abelian groups by their Fourier series.

5. Appendix: spectral theorem for unitary operators

Let \( V \) be a finite-dimensional complex vector space with a hermitian inner product \( \langle , \rangle \). A linear map \( T : V \to V \) is unitary if it preserves the inner product, in the sense that
\[
\langleTv, Tw\rangle = \langle v, w \rangle \quad \text{(for all } v, w \in V)\]
Thus, the adjoint \( T^* \) of a unitary operator \( T \) has the property
\[
\langle v, w \rangle = \langle Tv, Tw \rangle = \langle v, T^*w \rangle\]
Subtracting, \( \langle v, T^*w - w \rangle = 0 \) for all \( v \), so \( T^*T = T^2 \) for all \( w \). That is, unitary \( T \) is invertible, and \( T^* = T^{-1} \). This also shows that \( T T^* = TT^* \).

The inverse of a unitary operator is unitary, since
\[
\langle T^{-1}v, T^{-1}w \rangle = \langle T^*v, T^{-1}w \rangle = \langle v, TT^{-1}w \rangle = \langle v, w \rangle\]
Eigenvalues \( \lambda \) of a unitary operator \( T \) are of absolute value 1, since for a \( \lambda \)-eigenvector \( v \)
\[
\lambda \bar{\lambda} \langle v, v \rangle = \langle \lambda v, \lambda v \rangle = \langle Tv, T^2v \rangle = \langle v, v \rangle\]
In particular, eigenvalues \( \lambda \) are non-zero, and \( \lambda^{-1} = \bar{\lambda} \).

Given \( \lambda \in \mathbb{C} \), let
\[\lambda\text{-eigenspace of } T = V_\lambda = \{v \in V : Tv = \lambda \cdot v\}\]

[5.1] Theorem: The vectorspace is an orthogonal direct sum
\[
V = \bigoplus_\lambda V_\lambda \quad \text{(eigenspaces of unitary } T)\]

Proof: We grant ourselves the more elementary fact that, because \( V \) is finite-dimensional and \( \mathbb{C} \) is algebraically closed, there is at least one one eigenvalue \( \lambda \) and non-zero eigenvector \( v \) for \( T \). Thus, the \( \lambda \)-eigenspace \( V_\lambda \) is not \{0\}.

Now the unitariness is used, to set up an induction on dimension. We claim that \( T \) stabilizes the orthogonal complement
\[
V_\lambda^\perp = \{w \in V : \langle w, v \rangle = 0 \text{ for all } v \in V_\lambda\}\]
Indeed, for \( w \) in that orthogonal complement and \( v \in V_\lambda \),
\[
\langle Tw, v \rangle = \langle w, T^*v \rangle = \langle w, T^{-1}v \rangle = \langle w, \lambda^{-1}v \rangle = \lambda \langle w, v \rangle = 0 \quad \text{(for all } v \in V_\lambda)\]
The restriction of a unitary operator \( T \) to a \( T \)-stable subspace is obviously still unitary. By induction on the dimension of the vectorspace, \( V_\lambda^\perp \) is an orthogonal direct sum of \( T \)-eigenspaces: \( V_\lambda^\perp = \bigoplus_\mu V_\mu' \). Then
\[
V = V_\lambda \oplus \bigoplus_\mu V_\mu' \]
is the orthogonal direct sum decomposition of the whole space.