Definition: Let $\text{Hom}(A, B)$ denote the set of all maps from object $A$ to object $B$.

[3.1] Show that a finite set has a unique Hausdorff topology.

[3.2] Give an example of a non-Hausdorff space with a compact subset that is not closed.

[3.3] Prove that a colimit of coproducts is the coproduct of the colimits.

[3.4] Let $X = \varinjlim X_i$ be a limit of finite sets $X$. Let $p_i : X \to X_i$ be the projection to the $i^{th}$ limitand. Show that the topology on $X$ has a basis of sets $p_i^{-1}(x_i)$ for points $x_i \in X_i$.

[3.5] Map $\mathbb{Z} \to \mathbb{Z}/n$ by $z \to z + n\mathbb{Z}$ as usual. Show that $\mathbb{Z}$ is dense in the limit $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n$ under the induced map. Show that the sequence of factorials $n!$ goes to 0 in $\hat{\mathbb{Z}}$.

[3.6] Let $\{X_i : i \in I\}$ be objects indexed by a set $I$. Show that there is a natural isomorphism

$$\text{Hom}(\prod_i X_i, Y) \approx \prod_i \text{Hom}(X_i, Y)$$

and

$$\text{Hom}(Y, \prod_i X_i) \approx \prod_i \text{Hom}(Y, X_i)$$

[3.7] Let $\{X_i : i \in I\}$ be objects indexed by a poset $I$. Show that there is a natural isomorphism

$$\text{Hom}(\text{colim}_i X_i, Y) \approx \lim_i \text{Hom}(X_i, Y)$$

and

$$\text{Hom}(Y, \text{lim}_i X_i) \approx \lim_i \text{Hom}(Y, X_i)$$

[3.8] Let $\mu_n$ be the set of all $n^{th}$ roots of unity inside a fixed algebraic closure $\overline{\mathbb{Q}}$ of the rationals $\mathbb{Q}$. Let $\Omega$ be the field obtained by adjoining all $\mu_n$ to $\mathbb{Q}$, and let

$$\text{Gal}(\Omega/\mathbb{Q}) = \varprojlim_n \text{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$$

where the indices $n$ are ordered by divisibility. Prove that

$$\text{Gal}(\Omega/\mathbb{Q}) = \lim_n (\mathbb{Z}/n)^\times$$

[3.9] Let $\mathbb{F}_q$ denote a finite field with $q$ elements. Show that the Galois groups of the (separable) algebraic closure of $\mathbb{F}_q$ is naturally isomorphic to $\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n$, where the indices are ordered by divisibility.
(Infinite Galois theory) Let $k$ be a field. Let $I$ be the poset of finite Galois extensions $K$ of $k$, ordered by inclusion. Define the Galois group of the (separable) algebraic closure $\overline{k}$ of $k$ by

$$\text{Gal}(\overline{k}/k) = \lim_{\prod_{K \in I}} \text{Gal}(K/k)$$

as a \textit{topological} group. Show that the finite (separable) extensions of $k$ are in bijection with \textit{open} subgroups $H$ of $G$, by

$$H \leftrightarrow \text{fixed field of } H$$

Show that the not-necessarily finite (separable algebraic) extensions of $k$ are in bijection with \textit{closed} subgroups of $G$. 

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