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Exercises 4

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[4.1] Show that the map $[0,+\infty) \longrightarrow [0,1)$ of the half-line to a half-open interval $[0,1)$ given by

$$t \longrightarrow \frac{t}{t+1}$$

is a homeomorphism. But observe that the half-line with its usual metric is *complete*, while the half-open interval with the usual metric is *not* complete, since there is a Cauchy sequence converging to the point $1 \notin [0,1)$. *(Completeness is not a purely topological invariant)*

[4.2] Show that *any* metric inducing the product topology on a countable product of *compact* (for example, *finite*) topological spaces makes that product *complete*.

[4.3] Let X be a countable product of copies of (discretely topologized) copies of $\mathbf{Z}$, with metric

$$d(\{x_n\},\{y_n\}) = \sum_{n=1}^{\infty} 2^{-n}\, \frac{|x_n - y_n|}{1+|x_n-y_n|}$$

Prove that X is complete with this metric.

[4.4] Prove/observe that a topological space whose topology can be given by a metric (complete or not) must have a countable *local* basis at every point.

[4.5] *(Uncountable products are not metrizable)* Prove that an uncountable product of discrete two-point spaces is definitely *not* metrizable.

[4.6] Show that *uncountable* sums of positive real numbers cannot converge.

The following exercises are intended to be a retrospective warm-up to an earlier difficult exercise about a colimit of $\mathbf{R}/2^{-n}\mathbf{Z}$'s.

[4.7] As usual, $\mathbf{Z}[\frac{1}{2}]$ is the collection of rational numbers with denominators only involving powers of 2. Show that as abelian groups

$$\mathrm{colim}(\mathbf{Z}[\frac{1}{2}]/\mathbf{Z} \longrightarrow \mathbf{Z}[\frac{1}{2}]/\frac{1}{2}\mathbf{Z} \longrightarrow \mathbf{Z}[\frac{1}{2}]/\frac{1}{4}\mathbf{Z} \longrightarrow \mathbf{Z}[\frac{1}{2}]/\frac{1}{8}\mathbf{Z} \longrightarrow \ldots) \approx \{0\}$$

Equivalently, show that

$$\mathrm{colim}\left(\mathbf{Z}[\tfrac{1}{2}]/\mathbf{Z} \xrightarrow{\times 2} \mathbf{Z}[\tfrac{1}{2}]/\mathbf{Z} \xrightarrow{\times 2} \mathbf{Z}[\tfrac{1}{2}]/\mathbf{Z} \xrightarrow{\times 2} \cdots \right) \approx \{0\}$$

[4.8] Show that

$$\mathrm{colim}\left(\mathbf{Z}[\tfrac{1}{3}]/\mathbf{Z} \xrightarrow{\times 2} \mathbf{Z}[\tfrac{1}{3}]/\mathbf{Z} \xrightarrow{\times 2} \mathbf{Z}[\tfrac{1}{3}]/\mathbf{Z} \xrightarrow{\times 2} \cdots \right) \approx \mathbf{Z}[\frac{1}{3}]/\mathbf{Z}$$

[4.9] Show that

$$\mathbf{Q}/\mathbf{Z} \approx \coprod_p \mathbf{Z}[\frac{1}{p}]/\mathbf{Z}$$

[4.10] Show that

$$\operatorname{colim}\left(\mathbf{Q}/\mathbf{Z} \xrightarrow{\times 2} \mathbf{Q}/\mathbf{Z} \xrightarrow{\times 2} \mathbf{Q}/\mathbf{Z} \xrightarrow{\times 2} \cdots \right) \approx \mathbf{Z}_{(2)}/\mathbf{Z}$$

where $\mathbf{Z}_{(2)}$ is **Z localized at** 2, meaning that $\mathbf{Z}_{(2)}$ consists of all fractions a/b with $2 \not| b$.

[4.11] Show that colimits of products are not necessarily the product of the colimits. For example, the adeles $\mathbf{A}$ are definitely *not* the product $\mathbf{R} \times \Pi_p \mathbf{Q}_p$.

[4.12] Show that the 2-solenoid $(\mathbf{R} \times \mathbf{Q}_2)/\mathbf{Z}[\frac{1}{2}]^\Delta$ has *more* automorphisms, of the form

$$x + \mathbf{Z}[\frac{1}{2}]^\Delta \longrightarrow 2^n \cdot x + \mathbf{Z}[\frac{1}{2}]^\Delta$$

for integers n.

[4.13] Show that the ur-solenoid $\mathbf{A}/\mathbf{Q}$ has automorphism of the form

$$x + \mathbf{Q} \longrightarrow t \cdot x + \mathbf{Q}$$

for $t \in \mathbf{Q}^\times$.

[4.14] Show that the *discrete* subgroups of $\mathbf{R}$ are $\{0\}$ and subgroups $\mathbf{Z} \cdot r$ for $r \neq 0$.