[5.1] Verify that for a metric $d(,)$ on a space that $\delta(,) = d(,) / [1 + d(,)]$ is also a metric. The issue is the triangle inequality.

[5.2] Prove that $Z_p \cap Q$ is the localization

$$Z_{(p)} = \left\{ \frac{a}{b} \in Q : a, b \in Z, b \neq 0 \text{ mod } p \right\}$$

of $Z$ at $p$ (consisting of rational numbers without any factor of $p$ in their denominators).

[5.3] Using the mapping property characterization of the completion of a metric space, show that (as with the presumed inclusion of $Z_p$ in $Q_p$) the completion of a subset $E$ of a metric space $X$ has a natural inclusion into the completion of the larger space.

[5.4] Show that an odd integer $D$ has a square root in $Q_p$ if and only if it has a square root modulo 8.

[5.5] Using the exponential and logarithm functions, show that for a prime $p > 2$ the map

$$pZ_p \rightarrow 1 + pZ_p \subset Z_p^\times \quad \text{by} \quad x \mapsto e^x$$

is an isomorphism of topological groups (with group operation of multiplication in $1 + pZ_p$).

[5.6] Let $p > 2$ be prime. Show that the quotient group $Q_p^\times / (Q_p^\times)^2$ (non-zero $p$-adic numbers modulo squares) is a group isomorphic to $Z/2 \oplus Z/2$, and that, therefore, there are exactly 3 quadratic field extensions of $Q_p$.

[5.7] Show that there are exactly 7 quadratic field extensions of $Q_2$.

[5.8] Prove a slightly more complicated version of Hensel’s lemma, namely, that for a polynomial $f$ with coefficients in $Q_p$ and $x_1 \in Q_p$ such that

$$|f(x_1)|_p < |f'(x_1)|^2_p$$

prove that the recursion $x_{n+1} = x_n - f(x_n)/f'(x_n)$ gives a sequence converging to a root of $f(x) = 0$ in $Q_p$.

[5.9] Viewing the root-finding version of Hensel’s lemma as really just talking about linear factors of polynomials, formulate (and prove correctness of) a similar recursion (and, thereby, existence argument) for factoring a polynomial (into possibly higher degree factors) in $Q_p[x]$ if it factors modulo $pZ_p$.

[5.10] Determine the factorization of cyclotomic polynomials (defined recursively by)

$$\Phi_n(x) = \prod_{d|n} \frac{x^n - 1}{\Phi_d(x)}$$

(with $\Phi_1(x) = x - 1$) over $Q_p$. (Consider the case the $p$ does not divide $n$ first.)