(*November 19, 2005*)

Exercises 6

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[6.1] When, if ever, is the *box* topology on a *countable* product of metric spaces metrizable?

[6.2] Prove that $\mathbb{Z}_p$ is the unique maximal compact *subring* of $\mathbb{Q}_p$.

[6.3] Express $f(x) = \sum_{n\in\mathbb{Z}} 1/(x+in)^4$ (with $x \in \mathbb{R}$) as an elementary function.

[6.4] Can you identify

$$f(x) = \frac{1}{x} + \sum_{n\neq 0} \left(\frac{1}{x+n} - \frac{1}{n} \right)$$

as an elementary function? Is there any reason to think that it *is* elementary, without determining it explicitly?

[6.5] Let $\varphi : \mathbb{R}/\mathbb{Z} \longrightarrow \mathbb{C}^\times$ be a differentiable group homomorphism. Prove that for some $n \in \mathbb{Z}$

$$\varphi(x+\mathbb{Z}) = e^{2\pi i n x}$$

[6.6] Evaluate $1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \ldots$.

[6.7] Compute the *order* of $SL(n, \mathbb{Z}/N)$. (*Hint:* Start with N prime, then prime powers.)

[6.8] Prove that the natural maps $SL(n,\mathbb{Z}) \longrightarrow SL(n,\mathbb{Z}/N)$ and $GL(n,\mathbb{Z}) \longrightarrow GL(n,\mathbb{Z}/N)$ are surjective. (*Hint:* Start with N prime, then prime powers.)

[6.9] Let P be the group of upper-triangular matrices in $GL(n+1,\mathbb{C})$. How many orbits does P have on $\mathbb{P}^n$?

[6.10] Let P be the group of upper-triangular invertible n-by-n real matrices. Let $O(n)$ be the usual orthogonal group

$$O(n) = \{g \in GL(n,\mathbb{R}) \;:\; g^\top \cdot g = 1_n\}$$

where $g^\top$ is g-transpose and 1_n is the n-by-n identity matrix. Prove a special case of the **Iwasawa decomposition**, namely, that

$$GL(n,\mathbb{R}) = P \cdot O(n) = \{p \cdot k \;:\; p \in P,\; k \in O(n)\}$$