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Exercises 7

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[7.1] We have the vanishing $\int_{S^1} f(k)\,dk = 0$ for *non-trivial* (continuous) group homomorphisms $f : S^1 \longrightarrow \mathbb{C}^\times$, proven by changing variables $k \longrightarrow k+\delta$ for δ such that $f(\delta) = 0$. The map $z \longrightarrow (z+i)/(iz+1)$ sends the unit circle in $\mathbb{C}$ to the real line (plus the point at infinity). Do the change of coordinates to obtain

$$\int_{-\infty}^{+\infty} \left(\frac{x+i}{ix+1}\right) \frac{dx}{x^2+1} dx = 0 \qquad (\text{for } 0 \neq n \in \mathbb{Z})$$

[7.2] Show that all torsion elements in $SL_2(\mathbb{Z})$ are *conjugate* to one of $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$, $\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$, or $\begin{pmatrix} -1 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$,

[7.3] Prove that for $N > 3$ the principal congruence subgroup Γ_N has no *torsion* elements.

[7.4] Let G_N be the subgroup of $GL_n(\mathbb{Z})$ of elements which are congruent to 1_n (entrywise) modulo N. Let φ be the Euler totient function. Let μ be the maximum of all integers k such that $\varphi(k) \leq n$. Show that for $N > \mu$ the group G_N is torsion-free.

[7.5] Let L be a finitely-generated free $\mathbb{Z}$-module, and p a fixed prime, and let

$$X = \lim \left(\ldots \longrightarrow L/p^3L \longrightarrow L/p^2L \longrightarrow L/pL\right)$$

Show that

$$X \approx \mathbb{Z}_p \otimes_{\mathbb{Z}} L$$

as $\mathbb{Z}$-module (which also shows that X acquires a $\mathbb{Z}_p$-module structure). Next, exhibit natural *isomorphisms*

$$L/p^eL \approx X/p^eX$$

And then show that

$$\mathrm{Aut}_{\mathbb{Z}_p}(X) \approx \lim_e \mathrm{Aut}_{\mathbb{Z}}(L/p^eL)$$

[7.6] Let X be a real 2-by-2 matrix with trace 0. Show that the series for the *exponential*

$$e^X = \sum_{n\geq 0} \frac{X^n}{n!}$$

converges and gives an element of $SL_2(\mathbb{R})$. Find two trace-zero matrices X, Y such that

$$e^{X+Y} \neq e^X \cdot e^Y$$

Is this exponential map *surjective* to $SL_2(\mathbb{R})$?

[7.7] Let X be a 2-by-2 matrix with entries in $\mathbb{Q}_p$ and trace 0. Find conditions on X for the convergence of

$$e^X = \sum_{n\geq 0} \frac{X^n}{n!}$$

[7.8] For a positive integer n, the n^{th} **Hecke operator** T_n is an *averaging* operator defined on functions f on lattices Λ in $\mathbb{C}$, by

$$(T_n f)(\Lambda) = \sum_{\Lambda' \,:\, [\Lambda:\Lambda']=n} f(\Lambda')$$

Prove that for m, n relatively prime, $T_{mn} = T_m T_n$.

[7.9] Let $E_{2k}(\Lambda) = \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^{2k}}$ be the weight $2k$ **Eisenstein series** defined as a function of *lattices* $\Lambda \subset \mathbb{C}$. Show that E_{2k} is an eigenfunction for T_n and determine the eigenvalue.