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Exercises 10

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[10.1] Verify from the definition that the distributional derivative of the so-called *Heaviside function*

$$H(x) = \{ 0 \quad (\text{for x¡ 0})1 \quad (\text{for x¿ 0})$$

is the so-called *Dirac delta* (generalized) function defined as a functional by

$$\delta(f) = f(0) \qquad (\text{for } f \in C_c^\infty(\mathbb{R}))$$

[10.2] Extend the mean value theorem to distributions: Show that a distribution u such that $u' = 0$ is necessarily (integration against) a constant.

[10.3] Suppose that u is a distribution such that $xu = 0$, in the sense that $u(x \cdot f) = 0$ for all $f \in C_c^\infty(\mathbb{R})$. Show that u is a constant multiple of the Dirac delta.

[10.4] Since $\log|x|$ is locally integrable (even near 0), the linear functional

$$u(f) = \int_{\mathbb{R}} f(x)\,\log|x|\,dx \qquad (\text{for } f \in C_c^\infty(\mathbb{R}))$$

is a distribution. Show that its distributional derivative *can* be rewritten as the traditionally mysterious so-called *Cauchy principal value*

$$u'(f) = \lim_{\varepsilon \longrightarrow 0^+} \int_{|x|>\varepsilon} \frac{f(x)}{x}\,dx$$

[10.5] Describe in classical terms the distributional derivative of

$$f \longrightarrow \lim_{\varepsilon \longrightarrow 0^+} \int_{|x|>\varepsilon} \frac{f(x)}{x}\,dx$$

[10.6] Compute the Fourier transform of $xe^{-\pi x^2}$.

[10.7] Give an example of a tempered distribution which is *not* in any Sobolev space $H_s(\mathbb{R})$.

[10.8] Is the distribution

$$u(f) = \sum_{n=0}^\infty f^{(n)}(n)$$

tempered?

[10.9] Let $t \in \mathbb{R}^\times$ act on Schwartz functions $\mathscr{S} = \mathscr{S}(\mathbb{R})$ by $(f \circ t)(x) = f(tx)$. Show that the only tempered distribution u such that

$$u(f \circ t) = u(f) \qquad (\text{for all } t \in \mathbb{R}^\times,\ f \in \mathscr{S})$$

is (up to a constant) the Dirac delta $\delta(f) = f(0)$.