Modular forms and number theory exercises 03

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[0.0.1] Remark: The polynomials \( B_\ell(x) \) are essentially Bernoulli polynomials.

[0.0.2] Remark: To get zeta values, another similar approach is invocation of a Plancherel theorem\(^1\) that for a Fourier expansion \( f(x) \sim \sum_n a_n e^{2\pi i n x} \),

\[
\int_0^1 |f(x)|^2 \, dx = \sum_n |a_n|^2
\]

In this approach, only \( B_1(x), B_2(x), B_3(x) \) are needed, but we must integrate their squares. In the approach sketched above, we need \( B_2(x), B_4(x), B_6(x) \), but no squaring and no subsequent integration is needed.

[0.0.3] Remark: For hand computations with Bernoulli polynomials, it’s convenient to write them as polynomials in \( x - \frac{1}{2} \). For example, if \( B_\ell(x) \) is described as the integral of \( B_{\ell-1}(x) \) plus a constant (to meet the \( \int_0^1 B_\ell(x) \, dx = 0 \) condition), then the constant required in going from even index to odd index is simply 0. Further, for both parities, that constant becomes the value \( B_\ell(\frac{1}{2}) \).

[0.0.1]

\[
\int_0^1 B_\ell(x) \, dx = 0 \quad \text{and} \quad \int_0^1 B_\ell(x) e^{-2\pi i n x} \, dx = \frac{-1}{(2\pi i n)^\ell} \quad \text{for all } 0 \neq \ell \in \mathbb{Z}
\]

(b) Determine the first few \( B_\ell(x) \) explicitly, and use them to evaluate

\[
\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \ldots
\]

\[
\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \ldots
\]

\[
\frac{1}{1^6} - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \ldots
\]

(c) Determine the Euler factorization of

\[
L(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \frac{1}{6^s} + \ldots
\]

and obtain \( \zeta(2), \zeta(4), \zeta(6) \) as corollaries.

[0.0.1]

\[
\sum_{\ell} t^\ell B_\ell(x)
\]

[1] Proof of the Plancherel theorem depends upon first results on Hilbert spaces. This is a very mild prerequisite, easy to meet, as will be done shortly. The notion of Plancherel theorem makes sense in many, many situations, so is a device one can hope to rely upon systematically.

[2] As usual, a generating function for a list \( b_1, b_2, \ldots \) of numbers of other reasonable objects is an a priori infinite sum \( \sum_n b_n t^n \) or \( \sum_n b_n t^n/n! \) or other form, so that this infinite sum happens to have a convenient closed form, meaning a significantly simpler or more elementary expression. There are no guarantees of existence of simpler closed forms, and choice of details of a proposed infinite sum is an art in itself.