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Modular forms and number theory exercises 04

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[mfms 04.1] Let G be a finite abelian group, with dual group $\widehat{G}$. Show that $\widehat{G}$ *separates points* on G, in the sense that, given $x \neq y$ in G, there is a character $\chi \in \widehat{G}$ such that $f(x) \neq f(y)$.

[0.0.1] Remark: In the following, I had confounded two potentially correct assertions into a single incorrect one: representations induced from non-trivial characters on the *center* are *not* irreducible, but they are *isotypic*, meaning that they're direct sums of copies of a *single* representation.

[mfms 04.2] (**) Define a *finite Heisenberg group* by

$$H = \{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_q \}$$

with a fixed finite field $\mathbb{F}_q$. Verify that the center is

$$Z = \{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{F}_q \}$$

Verify that $H/Z \approx \mathbb{F}_q \oplus \mathbb{F}_q$, so is *abelian*. Thus, it is easy to identify many *one-dimensional* representations of H, by factoring through H/Z. The not-so-easy question is to show that, given a *non-trivial* central character $\omega : Z \to \mathbb{C}^\times$, there is a unique (isomorphism class of) irreducible representation(s) of H with central character ω. In particular,

$$\mathrm{Ind}_Z^H \omega = \text{isotypic (not irreducible!)} \qquad (\text{for } \omega \neq 1)$$

Given a non-trivial character ω on Z, extend it to

$$A = \text{a maximal abelian subgroup} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

by

$$\widetilde{\omega} : \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \omega(b)$$

Then

$$\mathrm{Ind}_A^H \widetilde{\omega} = \text{the unique irreducible with central character } \omega$$

Hint: It is possible to prove some of these things for *finite* Heisenberg groups by *counting*, if one knows that the (bi)regular representation of a finite (or even compact) group is the sum of all irreducibles, each occuring with multiplicity equal to its dimension. Use the elementary inequality

$$n_1^2 + \ldots + n_\ell^2 < (n_1 + \ldots + n_\ell)^2 \qquad (\text{for } n_i > 0, \text{ when } \ell > 1)$$