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Modular forms and number theory exercises 06

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[mfms 06.1] With the Dirichlet character

$$\chi(n) = \begin{cases} 1 & (\text{for } n = 1 \bmod 4) \\ -1 & (\text{for } n = 3 \bmod 4) \\ 0 & (\text{for } n = 2 \bmod 4) \end{cases}$$

use the Fourier expansions of the (essentially Bernoulli) polynomials

$$\begin{cases} x - \frac{1}{2} & = B_1(x) = \frac{-1}{2\pi i} \sum_{n\neq 0} \frac{e^{2\pi i n x}}{n} \\ \frac{1}{6}(x-\frac{1}{2})^3 - \frac{1}{24}(x-\frac{1}{2}) & = B_3(x) = \frac{-1}{(2\pi i)^3} \sum_{n\neq 0} \frac{e^{2\pi i n x}}{n^3} \\ \frac{1}{120}(x-\frac{1}{2})^5 - \frac{1}{144}(x-\frac{1}{2})^3 + \frac{7}{6!\cdot 8}(x-\frac{1}{2}) & = B_5(x) = \frac{-1}{(2\pi i)^5} \sum_{n\neq 0} \frac{e^{2\pi i n x}}{n^5} \end{cases}$$

evaluated at $x = \frac{1}{4}$ and $x = \frac{3}{4}$, to evaluate

$$L(1,\chi) = \tfrac{1}{1} - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \tfrac{1}{11} + \ldots$$

$$L(3,\chi) = \tfrac{1}{1^3} - \tfrac{1}{3^3} + \tfrac{1}{5^3} - \tfrac{1}{7^3} + \tfrac{1}{9^3} - \tfrac{1}{11^3} + \ldots$$

$$L(5,\chi) = \tfrac{1}{1^5} - \tfrac{1}{3^5} + \tfrac{1}{5^5} - \tfrac{1}{7^5} + \tfrac{1}{9^5} - \tfrac{1}{11^5} + \ldots$$

[mfms 06.2] (*) Evaluate $L(2,\chi)$ and $L(4,\chi)$ for

$$\chi(n) = \begin{cases} 1 & (\text{for } n = 1 \bmod 5) \\ -1 & (\text{for } n = 2 \bmod 5) \\ -1 & (\text{for } n = 3 \bmod 5) \\ 1 & (\text{for } n = 4 \bmod 5) \\ 0 & (\text{for } n = 0 \bmod 5) \end{cases}$$

using

$$\begin{cases} \frac{1}{2}(x-\frac{1}{2})^2 - \frac{1}{24} & = B_2(x) = \frac{-1}{(2\pi i)^2} \sum_{n\neq 0} \frac{e^{2\pi i n x}}{n^2} \\ \frac{1}{24}(x-\frac{1}{2})^4 - \frac{1}{48}(x-\frac{1}{2})^2 + \frac{7}{6!\cdot 8} & = B_4(x) = \frac{-1}{(2\pi i)^4} \sum_{n\neq 0} \frac{e^{2\pi i n x}}{n^4} \end{cases}$$