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Solenoids

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Circles are simple geometric objects, although the theory of *functions on them*, Fourier series, is complicated enough to have upset people in the 19^{th} century, and to have precipitated the creation of set theory by Cantor *circa* 1880. But we postpone the function theory to another time. Here, we consider circles and simple mappings among them.

Surprisingly, automorphism groups of *families* of circles connected by simple maps bring to light structures and objects invisible when looking at a single circle rather than the aggregate.

In particular, we discover p-adic *numbers* $\mathbb{Q}_p$ inside automorphism groups of families of circles, and even the *adeles* $\mathbb{A}$. These appearances are more important than *ad hoc* definitions as completions with respect to metrics, recalled later. That is, p-adic numbers and the adeles appear *inevitably* in the study of modestly complicated structures, and are parts of automorphism groups.

This discussion of automorphisms of families of circles is a warm-up to the more complicated situation of automorphisms of families of *higher-dimensional* objects [1] acted upon by *non-abelian* groups.

In all cases, an underlying theme is that when a group G acts [2] *transitively* [3] on a set X, then X is in bijection with G/G_x, where G_x is the isotropy subgroup [4] in G of a chosen base point x in X, by $gG_x \to gx$. The point is that such sets X are really *quotients of the group G*. Topological and other structures also correspond, under mild hypotheses. Isomorphisms $X \approx G/G_x$ are informative and useful, as seen later.

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1. *The 2-solenoid*

As reported by MacLane in his autobiography, around 1942 Eilenberg talked to MacLane (in Michigan) about families of circles related to each other by repeated *windings*, for example, double coverings, and about trying to understand the *limiting object*. We make this precise and repeat some of the discussion. The point is that a surprisingly complicated physical object is made from families of *circles*.

[1] The next example in mind is *modular curves*, which are two-dimensional. Their definition is considerably more complicated than that of circles, and requires more preparation.

[2] Of course, the spirit of the notion of *action* of G on X is that G moves around elements of the set X. But a little more precision is needed. Recall that an action of G on a set X is a map $G \times X \to X$ such that $1_G \cdot x = x$ for all $x \in X$, and $(gh)x = g(hx)$ for $g, h \in G$ and $x \in X$.

[3] Recall that a group G acts *transitively* on a set X if, for all $x, y \in X$, there is g in G such that $gx = y$.

[4] Recall that the *isotropy subgroup* G_x of a point x in a set X on which G acts is the subgroup of G *fixing* x, that is, G_x is the subgroup of $g \in G$ such that $gx = x$.

As a model for the circle S^1 we take $S^1 = \mathbb{R}/\mathbb{Z}$.[5] Eilenberg (and MacLane) considered a family of circles and maps

$$\ldots \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z} \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z} \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z}$$

where each circle mapped to the next by *doubling* itself onto the smaller circle.[6] More precisely, this is literally multiplication by 2 on the quotients $\mathbb{R}/\mathbb{Z}$, namely

$$x + \mathbb{Z} \to 2x + \mathbb{Z}$$

for $x \in \mathbb{R}$. Since $2\mathbb{Z} \subset \mathbb{Z}$ this is well-defined. That is, each circle is a *double cover* of the circle to its immediate right in the sequence. This sequence of circles with doubling maps is the **2-solenoid**.[7] We might ask *what is the limiting object*

$$??? \quad \ldots \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z} \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z} \xrightarrow{\times 2} \mathbb{R}/\mathbb{Z}$$

Part of the issue is to say what we might *mean* by this question.

A different but topologically equivalent model is a little more convenient for our discussion. Consider the sequence

$$\ldots \xrightarrow{\varphi_{43}} \mathbb{R}/8\mathbb{Z} \xrightarrow{\varphi_{32}} \mathbb{R}/4\mathbb{Z} \xrightarrow{\varphi_{21}} \mathbb{R}/2\mathbb{Z} \xrightarrow{\varphi_{10}} \mathbb{R}/\mathbb{Z}$$

where each map $\varphi_{n,n-1} : \mathbb{R}/2^n\mathbb{Z} \to \mathbb{R}/2^{n-1}\mathbb{Z}$ is induced from the identity map on $\mathbb{R}$ in the diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{mod}\ 2^n"'] & \mathbb{R} \arrow[d, "\mathrm{mod}\ 2^{n-1}"] \\
\mathbb{R}/2^n\mathbb{Z} \arrow[r, dashed, "\varphi_{n,n-1}"] & \mathbb{R}/2^{n-1}\mathbb{Z}
\end{tikzcd}$$

That is, this is the map

$$\varphi_{n,n-1} : x + 2^n\mathbb{Z} \to x + 2^{n-1}\mathbb{Z}$$

This second model has the advantage that the maps $\varphi_{n,n-1}$ are locally distance-preserving on the circles. In the first model each map *stretches* the circle by a factor of 2. In the second model as we move to the left in the sequence of circles the circles get larger. Also, in the second model there is the single copy of $\mathbb{R}$ lying over (or *uniformizing*) all the circles.

Again we would like to ask *what is the limit of these circles?* Note that this use of *limit* is ambiguous, and we cannot be sure *a priori* that there is any potential sense to be made of this. Presumably we expect the limit to be a *topological space*.

[5] One could also take the unit circle in $\mathbb{C}$. The map $x \to e^{2\pi i x}$ mapping $\mathbb{R} \to \mathbb{C}$ factors through the quotient $\mathbb{R}/\mathbb{Z}$, showing that these two things are the same.

[6] The integer 2 could be replaced with other integers, and, for that matter, the *sequence* $2, 2, 2, \ldots$ could be replaced with other sequences of integers. Qualitatively, these other choices give similar things, though the details are significantly different.

[7] The name is by analogy with the wiring in electrical motors and inductance circuits, where things are made by repeated winding. Since the limiting object here is allegedly created by repeated *unwinding*, it might be more apt to call it an *anti*-solenoid.

[1.0.1] Remark: A plausible, naive guess would be that since $\bigcap_n 2^n\mathbb{Z} = \{0\}$, that perhaps the limit is $\mathbb{R}/\{0\} \approx \mathbb{R}$. It is not. [8]

A precise version of the question is the following. Consider

$$\cdots \xrightarrow{\varphi_{n+1,n}} X_n \xrightarrow{\varphi_{n,n-1}} X_{n-1} \xrightarrow{\varphi_{n-1,n-2}} \cdots \xrightarrow{\varphi_{21}} X_1 \xrightarrow{\varphi_{10}} X_0$$

with topological spaces X_i and continuous **transition maps** $\varphi_{i,i-1}$. The **(projective) limit** X of the X_n, written

$$X = \lim_i X_i \qquad \text{(dangerously suppressing reference to transition maps } \varphi_{i,i-1})$$

is a topological space X and maps $\varphi_n : X \to X_n$ **compatible with** the transition maps $\varphi_{n,n-1} : X_n \to X_{n-1}$ in the sense that

$$\varphi_{n-1} = \varphi_{n,n-1} \circ \varphi_n$$

and such that, for any *other* space Z with maps $f_n : Z \to X_n$ **compatible with** the maps $\varphi_{n,n-1}$ (that is, $f_{n-1} = \varphi_{n,n-1} \circ f_n$), there is a unique $f : Z \to X$ through which all the maps f_n factor. That is, in pictures, first, all the (curvy) triangles commute in

$$\begin{tikzcd} X \arrow[rr, bend left, "\varphi_1"] \arrow[rrr, bend left, "\varphi_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \end{tikzcd}$$

and, for all families of maps $f_i : Z \to X_i$ such that all triangles commute in

$$\begin{tikzcd} \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\ \cdots & & \\ Z \arrow[uur, "f_1"] \arrow[uurr, "f_0"'] & & \end{tikzcd}$$

there is a unique map $f : Z \to X$ such that all triangles commute in

$$\begin{tikzcd} X \arrow[rr, bend left, "\varphi_1"] \arrow[rrr, bend left, "\varphi_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\ & \cdots & & \\ & Z \arrow[uul, dashed, "f"] \arrow[uur, "f_1"] \arrow[uurr, "f_0"'] & & \end{tikzcd}$$

[1.0.2] Remark: Note that the definition of the limit definitely *does* depend on the *transition maps* among the objects of which we take the limit, not just on the *objects*.

As usual with mapping-property definitions:

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[8] Even though the limit of spaces $\mathbb{R}/n\mathbb{Z}$ fails to be $\mathbb{R}$ in several ways, this flawed notion *can* be used as a heuristic to see how Fourier inversion on $\mathbb{R}$ could be inferred from inversion for Fourier series.

[1.0.3] Theorem: If a (projective) limit exists it is unique up to unique isomorphism. *(Proof in appendix: it works for the usual abstract reasons.)*

A little more concretely, we can prove *existence* of limits from existence of *products* (at least for topological spaces):

[1.0.4] Proposition: A limit X (and maps $\varphi_i : X \to X_i$) of a family

$$\ldots \xrightarrow{\varphi_{n+1,n}} X_n \xrightarrow{\varphi_{n,n-1}} X_{n-1} \xrightarrow{\varphi_{n-1,n-2}} \ldots \xrightarrow{\varphi_{21}} X_1 \xrightarrow{\varphi_{10}} X_0$$

is a subset X (with the subset topology)[9] of the product[10] $Y = \prod_i X_i$ (with projections $p_i : Y \to X_i$) on which the projections are compatible with the transition maps $\varphi_{i,i-1}$, that is,

$$X = \{x \in Y \ : \ \varphi_{i,i-1}(p_i(x)) = p_{i-1}(x) \text{ for all } i\}$$

with maps φ_i obtained by restriction of the projection maps p_i from the whole product to X, namely

$$\varphi_i = p_i|_X : X \to X_i$$

Proof: First, let $j : X \to Y$ be the inclusion of X into Y, and let $\varphi_i : X \to X_i$ be the restriction of the projection $p_i : Y \to X_i$ to the subset X of the product Y. That is,

$$\varphi_i = p_i \circ j$$

Then, before thinking about any other space Z and other maps, we do have a diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[rrr, bend left=50, "\varphi_1"] \arrow[rrrr, bend left=60, "\varphi_0"] & Y \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left=40, "p_0"] & \ldots & X_1 & X_0 \end{tikzcd}$$

with commuting (curvy) *solid* triangles. While the maps from Y do *not* respect the transition maps $\varphi_{i,i-1} : X_i \to X_{i-1}$, by the very definition of the subset X of Y, the restrictions $\varphi_i = p_i \circ j$ of the projections p_i to X *do* respect the transition maps. Thus, the *solid* triangles commute in the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[rrr, bend left=50, "\varphi_1"] \arrow[rrrr, bend left=60, "\varphi_0"] & Y \arrow[rr, bend left, dotted, "p_1"] \arrow[rrr, bend left=40, dotted, "p_0"] & \ldots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \end{tikzcd}$$

but *not* necessarily any triangle involving dotted arrows.

[9] The *subset topology* on a subset X of a topological space Y can be characterized as the topology on X such that the inclusion map $j : X \to Y$ is continous, and such that every continuous map $f : Z \to Y$ from another space Z such that $f(Z) \subset X$ factors through the inclusion. That is, there a continuous $F : Z \to X$ such that $f = i \circ F$. This does not prove existence. Also, one can show that the subspace topology is the coarsest topology on X such that the inclusion $X \to Y$ is continuous. Finally, the *construction* of this topology, which proves *existence*, is that a set U in X is open if and only if there exists an open V in Y such that $U = X \cap V$.

[10] By now we know that the usual product can be characterized *intrinsically*, and that intrinsic characterization is all we use in this proposition.

Now consider another space Z. By the mapping properties of the product, for *any* collection of maps $f_i : Z \to X_i$ (not only those meeting the compatibility condition $\varphi_{i,i-1} \circ f_i = f_{i-1}$) there is a unique $F : Z \to Y$ through which all the projections $p_i : Y \to X_i$ factor. That is, we have a unique F such that the triangles commute in the diagram

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[rrr, bend left=50, "\varphi_1"] \arrow[rrrr, bend left=60, "\varphi_0"] & Y \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left=40, "p_0"] & \cdots & X_1 & X_0 \\
& & Z \arrow[ul, dashed, "F"] \arrow[ur, "f_1"'] \arrow[urr, "f_0"'] \arrow[u, phantom, "\cdots"] & &
\end{tikzcd}$$

Note that we cannot include the transition maps in the diagram since the projections $p_i : Y \to X_i$ do not respect them. But, since the maps f_i *are* compatible with the maps $\varphi_{i,i-1}$, we could suspect that the image $F(Z) \subset Y$ is a smaller subset of the product Y. Indeed, for $z \in Z$, using the compatibility

$$p_{i-1}(F(z)) = f_{i-1}(z) = \varphi_{i,i-1}(f_i(z)) = \varphi_{i,i-1}(p_i(F(z)))$$

we see that $F(Z) \subset X$, as claimed. That is, F factors through the inclusion map $j : X \to Y$, and the composites $p_i \circ F$ factor through $j : X \to Y$, giving a picture with commuting *solid or dashed (but not dotted)* triangles

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[rrr, bend left=50, "\varphi_1"] \arrow[rrrr, bend left=60, "\varphi_0"] & Y \arrow[rr, dotted, bend left, "p_1"] \arrow[rrr, dotted, bend left=40, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
& & Z \arrow[ull, dashed, "f"] \arrow[ul, dashed, "F"] \arrow[ur, "f_1"'] \arrow[urr, "f_0"'] \arrow[u, phantom, "\cdots"] & &
\end{tikzcd}$$

(Again, the projections from Y do not respect the transition maps.) That is, with the compatibility conditions, the maps from Z *do* factor through the subset X of the product. ///

This general argument gives some surprising *qualitative* information about projective limits:

[1.0.5] Corollary: The projective limit of a family X_i of *compact*[11] Hausdorff spaces is *compact.*

[1.0.6] Remark: In particular, the (projective) limit of circles is *compact*, since circles (with their usual topologies) are compact. In particular, this limit cannot be the non-compact $\mathbb{R}$.

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[11] We need a better definition of *compact* than the metric-space definition that *every sequence contains a convergent subsequence*. Instead, we need the definition that both applies to general topological spaces *and* is more useful. That is, first in words, a set E inside a topological space is *compact* if *every open cover admits a finite subcover*. That is, for $E \subset \bigcup_{i\in I} U_i$ with opens U_i, there is a finite subset I_o of I such that still $E \subset \bigcup_{i\in I_o} U_i$. It is not obvious that this definition is superior to the sequence definition.

Proof: *(of corollary)* The essence of the argument is that products of compact spaces are compact, by Tychonoff, and closed subspaces of compact Hausdorff[12] spaces are compact. Thus, the product $Y = \prod_i X_i$ is compact. The compatibility conditions $\varphi_{i,i-1}(p_i(x)) = p_{i-1}(x)$ are *closed* conditions in the sense that

$$\{x \in Y : \varphi_{i,i-1}(p_i(x)) = p_{i-1}(x)\} = \text{closed set in } Y$$

since the maps p_i and $\varphi_{i,i-1}$ are continuous and since the spaces X_i are assumed *Hausdorff.* [13] [14] The intersection of an arbitrary family of closed sets is closed, [15] so the (projective limit) X of points x meeting this condition for *all* i, is closed. And in a compact space Y, closed subsets are compact. [16] ///

[1.0.7] **Remark:** By paraphrasing the assertion of the proposition, we now have a concrete (if not perfectly useful) model of the limit X in a diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "\varphi_2"] \arrow[rrr, bend left, "\varphi_1"] \arrow[rrrr, bend left, "\varphi_0"] & \cdots \arrow[r, "\varphi_{32}"] & X_2 \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0
\end{tikzcd}$$

with spaces indexed by non-negative integers, namely, the collection of all sequences $x_o, x_1, x_2, \ldots$ such that the transition maps $\varphi_{n,n-1}$ map them to each other, that is,

$$\varphi_{n,n-1}(x_n) = x_{n-1}$$

for all indices n. This follows from the usual model of the *product* as Cartesian product, which for countable products can be written as the collection of *all* sequences $x_o, x_1, x_2, \ldots$ with $x_i \in X_i$. We may choose to write a *compatible* family of elements as

$$\ldots \to x_3 \to x_2 \to x_1 \to x_0$$

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[12] Recall that a topological space is *Hausdorff* if any two points have disjoint neighborhoods. It is useful to know that Z is Hausdorff if and only if the diagonal $Z^\Delta = \{(z,z) \in Z \times Z : z \in Z\}$ is *closed* in $Z \times Z$. Indeed, for Z Hausdorff, points $x \neq y$ in Z have disjoint neighborhoods U and V. Then $U \times V$ is open in the product topology in $Z \times Z$, contains $x \times y$, and since $U \cap V = \phi$ the set $U \times V$ does not meet the diagonal $\{(z,z) : z \in Z\}$ in $Z \times Z$. Thus, the diagonal is the complement of the union of all such opens $U \times V$, so is closed. The converse reverses the argument: for closed diagonal, given $x \neq y$ in Z, there is an open $U \times V$ containing $x \times y$ and not meeting the diagonal, since the product topology has sets $U \times V$ as a basis. Since $U \times V$ does not meet the diagonal, U and V are disjoint neighborhoods of x, y in Z.

[13] As a critical auxiliary point, we should note that for any topological space X the diagonal imbedding $\delta : X \to X \times X$ by $\delta(x) = (x,x)$ is a homeomorphism (topological isomorphism) to the image, with the subspace topology. Certainly δ is a set bijection. For a neighborhood U of x in X, the open $U \times U$ in $X \times X$ meets $\delta(X)$ at $\delta(U)$. On the other hand, given opens U, V in X, the basis open $U \times V$ in $X \times X$ meets $\delta(X)$ in $\delta(U \cap V)$, and, indeed, $U \cap V$ is open. Thus, the images by δ of opens are open, and vice-versa.

[14] Characterizing Hausdorff-ness by the closed-ness of the diagonal is useful to show that for continuous maps $f : X \to Z$ and $g : X \to Z$ with Z Hausdorff, the set $\{x \in X : f(x) = g(x)\}$ is *closed*, as follows. The map $(f \times g)(x,y) = f(x) \times g(y)$ from $X \times X$ to $Z \times Z$ is continuous, that is, inverse images of opens are open. Then inverse images of closed sets are closed, and the inverse image of Z^Δ under $f \times g$ is closed. The intersection of the diagonal with the inverse image of Z^Δ by $f \times g$ is $\{(x,x) : f(x) = g(x)\}$. Closed-ness in $X \times X$ gives closedness in the diagonal (with the subspace topology), and we just noted that the diagonal is homeomorphic (topologically isomorphic) to X.

[15] That an arbitrary intersection of closed sets is closed is equivalent to the defining property that an arbitrary union of open sets is open, since a set is closed if and only if its complement is open.

[16] This important fact is easy to prove: let E be a closed subset of a compact space Y, and let $\{U_i : i \in I\}$ be an open cover of E. Let $U = Y - E$. Then $\{U_i : i \in I\} \cup \{U\}$ is a cover of the entire space Y. By the compactness of Y, there is a finite subcover $U_1, \ldots, U_n, U$. (If $E \neq Y$ the subcover must use U.) Then $U_1, \ldots, U_n$ is a finite cover of E.

This description of the limit as a set of sequences is *deficient* in several regards (for example, it does not tell us a *topology*), but it is occasionally useful, certainly as a heuristic.

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2. *Automorphisms of solenoids*

Even without trying to imagine what meaning to attach to a solenoid X or other limit object, we can *directly* make sense of *automorphisms* of X by looking at automorphisms [17] of the *diagram*. Then, with a large-enough group G of automorphisms to act *transitively*[18] on X, we can write X as a *quotient*

$$X \approx G/G_x = \{G_x\text{-cosets in } G\} = \{gG_x : g \in G\}$$

of G, where G_x is the *isotropy subgroup*[19] (in G) of a point x in X.

One virtue of identifying automorphisms $X \approx G/G_x$ is that this identification might be done piece-by-piece, identifying subgroups of the whole group, then assembling them at the end. And it is important to note that this is an isomorphism of G-**spaces**, meaning (topological) spaces A, B on which G acts continuously. As expected, a map of G-spaces is a set map $\psi : A \to B$ such that

$$\psi(g \cdot a) = g \cdot \psi(a)$$

for $a \in A$, $g \in G$, where on the left the action is of G on A, and on the right it is the action on B.

True, the solenoid is itself a group already, being a projective limit of groups, so this approach might seem silly. However, we can present the solenoid as a quotient of *more familiar* (and simpler) objects. In any case, since we'll consider an *abelian* group G of automorphisms of the solenoid, any group quotient G/G_x is again a group, [20] and, incidentally, G_x and the quotient are independent of x.

Thus, even without thinking of projective limits, one kind [21] of **automorphism** f of the 2-solenoid is a collection of maps $f_n : \mathbb{R}/2^n\mathbb{Z} \to \mathbb{R}/2^n\mathbb{Z}$ such that all squares commute in the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_{43}"] & \mathbb{R}/8\mathbb{Z} \arrow[r, "\varphi_{32}"] \arrow[d, "f_3"] & \mathbb{R}/4\mathbb{Z} \arrow[r, "\varphi_{21}"] \arrow[d, "f_2"] & \mathbb{R}/2\mathbb{Z} \arrow[r, "\varphi_{10}"] \arrow[d, "f_1"] & \mathbb{R}/\mathbb{Z} \arrow[d, "f_0"] \\
\cdots \arrow[r, "\varphi_{43}"] & \mathbb{R}/8\mathbb{Z} \arrow[r, "\varphi_{32}"] & \mathbb{R}/4\mathbb{Z} \arrow[r, "\varphi_{21}"] & \mathbb{R}/2\mathbb{Z} \arrow[r, "\varphi_{10}"] & \mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

Without being too extravagant [22] we want to think of some obvious families of maps f_n. Since all our circles are quotients of $\mathbb{R}$ in a compatible fashion, we can certainly create a simple sort of family of maps f_n by letting $r \in \mathbb{R}$ act, by

$$f_n(x_n + 2^n\mathbb{Z}) = x_n + r + 2^n\mathbb{Z}$$

[17] Note that this discussion is different from the argument that objects defined by mapping properties have no endomorphisms that leave the other objects unmoved. Here we *are moving* the objects in the diagram.

[18] Again, for a group to act *transitively* means that G moves any point of X to any other point, that is, for $x, y \in X$ there is $g \in G$ such that $gx = y$.

[19] Again, with a group G acting on a set X, the *isotropy subgroup* G_x of an element $x \in X$ is the subgroup *not moving* x, that is, $G_x = \{g \in G : gx = x\}$. It is straightforward to see that this is a sub*group* of G, not merely a sub*set*.

[20] We will attend to topological details slightly later.

[21] It is not *a priori* clear that any useful collection of maps would necessarily send each $\mathbb{R}/2^n\mathbb{Z}$ to itself, but this will suffice for now.

[22] But sometimes extravagance can have a simplicity that is hard to achieve otherwise.

with the same real number r for every index. [23]

[2.0.1] Remark: We are neglecting continuity, but will return to this point when we recapitulate this discussion of automorphisms in a form that is better suited to discussion of the topology. This copy of $\mathbb{R}$ *does* act continuously. One may verify that it is *not* transitive, and that the isotropy groups in $\mathbb{R}$ of points in the solenoid are *trivial*. Thus, overlooking the failure of the action to be transitive, one might naively imagine that the limit *is* a copy of $\mathbb{R}$. True, the orbit $\mathbb{R}\cdot x$ of any given point is *dense*[24] in the solenoid, but it is not *closed*. [25]

Another relatively simple family of maps is created by taking a sequence of *integers* y_n and maps

$$f_n(x_n + 2^n\mathbb{Z}) = x_n + y_n + 2^n\mathbb{Z}$$

and *requiring* that the sequence y_n be chosen so that the squares in the diagram commute. That is, we must have

$$(x_n + y_n + 2^n\mathbb{Z}) + 2^{n-1}\mathbb{Z} = x_{n-1} + y_{n-1} + 2^{n-1}\mathbb{Z}$$

Since already

$$(x_n + 2^n\mathbb{Z}) + 2^{n-1}\mathbb{Z} = x_{n-1} + 2^{n-1}\mathbb{Z}$$

it is necessary and sufficient that

$$(y_n + 2^n\mathbb{Z}) + 2^{n-1}\mathbb{Z} = y_{n-1} + 2^{n-1}\mathbb{Z}$$

That is, the compatible sequence of integers y_n gives an element in another projective limit, the **2-adic integers**[26] $\mathbb{Z}_2$.

$$\cdots \xrightarrow{\text{mod } 8} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\text{mod } 4} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\text{mod } 1} \mathbb{Z}/\mathbb{Z}$$

Each of the limit objects is *finite*, so certainly *compact*. Thus, this projective limit is *compact*, whatever other features it may have.

Still without worrying about the topology, we claim

[2.0.2] Proposition: The product group $\mathbb{R}\times\mathbb{Z}_2$ acts transitively on the 2-solenoid. The point $\to 0 \to 0 \to 0$ in the solenoid has isotropy group which is the diagonally imbedded copy of the integers

$$\mathbb{Z}^\Delta = \{(\ell, -\ell) \in (\mathbb{Z}\times\mathbb{Z}) \subset \mathbb{R}\times\mathbb{Z}_2 \;:\; \ell\in\mathbb{Z}\}$$

Proof: Given a compatible family

$$\ldots \to x_3 + 8\mathbb{Z} \to x_2 + 4\mathbb{Z} \to x_1 + 2\mathbb{Z} \to x_0 + \mathbb{Z}$$

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[23] And the $x_i \in \mathbb{R}/2^i\mathbb{Z}$ are chosen compatibly in the first place, that is, such that $(x_i + 2^i\mathbb{Z}) + 2^{i-1}\mathbb{Z} = x_{i-1} + 2^{i-1}\mathbb{Z}$, for all indices i.

[24] Recall that a subset E of a topological space X is *dense* if every non-empty open set in X has non-empty intersection with E.

[25] This highly-wound copy of $\mathbb{R}$ may be the thing that earned the name *solenoid*.

[26] Replacing 2 by another prime p throughout gives a p-solenoid and p-adic integers $\mathbb{Z}_p$. This approach is not the most conventional way to present the p-adic integers, but *does* illustrate the role that $\mathbb{Z}_p$ plays in situations that are not obviously number-theoretic. We will review a more conventional description of $\mathbb{Z}_p$ later, for comparison.

of elements $x_n + 2^n\mathbb{Z} \in \mathbb{R}/2^n\mathbb{Z}$, act by $r \in \mathbb{R}$ as above such that $x_0 + r = 0 \in \mathbb{R}/\mathbb{Z}$. Since the x_n's are compatible, it must be that $(r+x_1) \bmod 1 = (x_0+r) = 0$, $(x_2+r) \bmod 2 = (x_1+r)$, $(x_3+r) \bmod 4 = x_2+r$, and so on. That is, every $x_n + r \in \mathbb{Z}$, and the sequence $y_n = x_n + r$ gives a compatible family

$$\ldots \to y_3 + 8\mathbb{Z} \to y_2 + 4\mathbb{Z} \to y_1 + 2\mathbb{Z} \to y_0 + \mathbb{Z}$$

which gives an element in $\mathbb{Z}_2$. That is, the further action by $-y_n$ on the solenoid will send every element to

$$(x_n + r) - y_n = (x_n + r) - (x_n + r) = 0$$

This proves the transitivity.

To determine the isotropy group of a point, suppose that r is a real number and the y_n is an integer modulo 2^n, such that the 0-element

$$\ldots 0 \to 0 \to 0 \to 0$$

is mapped to itself. That is, require that

$$0 + r + y_n \in 0 + 2^n\mathbb{Z}$$

for all n. First, this implies that $r \in \mathbb{Z}$. Then y_n, which is only determined modulo 2^n anyway, is completely determined modulo 2^n by

$$y_n + 2^n\mathbb{Z} = -r + 2^n\mathbb{Z}$$

That is, $y_n = -r \bmod 2^n$. And these conditions are visibly *sufficient*, as well, to fix the 0. Thus, the isotropy group truly is the diagonal copy of $\mathbb{Z}$. ///

[2.0.3] Corollary: (Still not worrying about the topology) the 2-solenoid is isomorphic to the quotient

$$(\mathbb{R} \times \mathbb{Z}_2)/\mathbb{Z}^\Delta$$

Proof: Notably ignoring the topology, whenever a group G acts transitively on a set X containing a chosen element x, there is a bijection

$$X \longleftrightarrow G/G_x = \{gG_x \ : \ g \in G\}$$

by

$$gx \longleftrightarrow gG_x$$

A map from G to X by $g \to gx$ is a *surjection*, since G is transitive. This map factors through G/G_x and is *injective*, since $gx = hx$ if and only if $h^{-1}gx = x$, if and only if $h^{-1}g \in G_x$, if and only if $gG_x = hG_x$. ///

[2.0.4] Remark: We need a somewhat better set-up to keep track of the topologies.

---

3. *A cleaner viewpoint*

Having run through an informative heuristic about the structure of the solenoid as a quotient G/G_x, we can redo things more elegantly, and *not* lose sight of the topological features of the situation.

First, in any projective limit, families of maps[27] $f_n : X_n \to X_n$ such that all squares commute in

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_{32}"] & X_2 \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
\cdots \arrow[r, "\varphi_{32}"] & X_2 \arrow[u, "f_2"] \arrow[r, "\varphi_{21}"] & X_1 \arrow[u, "f_1"] \arrow[r, "\varphi_{10}"] & X_0 \arrow[u, "f_0"]
\end{tikzcd}$$

[27] Again, *maps* here are continuous maps, but the arguments do not use this explicitly.

do give rise to a map $f : X \to X$ of the projective limit $X = \lim_n X_n$ to itself, as follows. Again, from the definition of the projective limit X of the X_n, to give a map $F : Z \to X$ is to give a compatible family of maps $F_n : Z \to X_n$, meaning that all triangles commute in

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
\cdots & & \\
Z \arrow[uur, "F_1"] \arrow[uurr, "F_0"'] & &
\end{tikzcd}$$

This induces a unique map $F : Z \to X$, making a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
 & \cdots & & \\
 & Z \arrow[uul, dashed, "F"] \arrow[uur, "F_1"] \arrow[uurr, "F_0"'] & &
\end{tikzcd}$$

In particular, for a compatible family of maps $f_n : X_n \to X_n$ we can take $Z = X$ and $F_n = f_n \circ p_n$, giving a commutative

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
 & \ldots & & \\
X \arrow[uurr, dashed, "F_1"] \arrow[uurrr, dashed, "F_0"'] \arrow[rr, bend right, "p_1"'] \arrow[rrr, bend right, "p_0"'] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[uu, "f_1"] \arrow[r, "\varphi_{10}"] & X_0 \arrow[uu, "f_1"]
\end{tikzcd}$$

which then yields a unique $F : X \to X$ in a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
 & \ldots & & \\
X \arrow[uu, dotted, "F"] \arrow[uurr, dashed, "F_1"] \arrow[uurrr, dashed, "F_0"'] \arrow[rr, bend right, "p_1"'] \arrow[rrr, bend right, "p_0"'] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[uu, "f_1"] \arrow[r, "\varphi_{10}"] & X_0 \arrow[uu, "f_1"]
\end{tikzcd}$$

That is, automorphisms of *diagrams* (in the sense of the previous section) do give automorphisms of the projective limit objects attached to the diagrams.

We also observe that we can identify *points* in the projective limit as *compatible sequences*

$$\ldots \to x_3 \to x_2 \to x_1 \to x_0$$

with $x_n \in X_n$ (and the compatibility $\varphi_{n,n-1}(x_n) = x_{n-1}$) *without* using the Cartesian product model of the product and identifying the projective limit inside that Cartesian product. To do so, recall the trick that for any set Y and for $\{s\}$ a set with a single element, we have a natural bijection

$$\mu_Y : Y = \{\text{elements of } Y\} \longleftrightarrow \{\text{maps } \{s\} \to Y\}$$

by

$$\mu_Y : y \to f \text{ with } f(s) = y$$

These maps μ_Y are *natural* in the precise sense that for a set map $f : Y \to Z$, we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\mu_Y"] \arrow[d, "f"'] & \{\text{maps } \{s\} \to Y\} \arrow[d, "f\circ -"] \\ Z \arrow[r, "\mu_Z"] & \{\text{maps } \{s\} \to Z\} \end{tikzcd}$$

where $f \circ -$ is *post* composition with f, that is $\varphi \to f \circ \varphi$. And since maps to a projective limit $X = \lim X_n$ are given exactly by compatible family of maps to the X_n, maps of $S = \{s\}$ to X are given by compatible families of maps to the X_n as in

$$\begin{tikzcd} X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left=40, "p_0"] & \ldots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\ & \{s\} \arrow[ul, dashed] \arrow[ur] \arrow[urr] & & \end{tikzcd}$$

That is, *elements* of X are given by compatible families of elements of the X_n, as claimed. This will be useful in proving *transitivity* of a group action.

A **topological group** is a group G which has a topology such that multiplication $g \times h \to gh$ and inversion $g \to g^{-1}$ are continuous maps $G \times G \to G$ and $G \to G$, *and* G is *locally compact*[28] and *Hausdorff.* [29] Further, it is often necessary or wise to require that a topological group have a *countable basis.* [30] An action of a topological group G on a topological space X is **continuous** if the map

$$G \times X \to X \quad \text{by} \quad g \times x \to gx \text{ is continuous}$$

Now we see how to get an action of a topological group G on a projective limit.

[3.0.1] Claim: Let $a_n : G \times X_n \to X_n$ be continuous group actions of a topological group G on topological spaces X_n, and suppose that these actions are *compatible* in the sense that squares commute in the diagram

$$\begin{tikzcd} \cdots \arrow[r, "\varphi_{32}"] & X_2 \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\ \cdots \arrow[r, "\mathrm{id}_G\times\varphi_{32}"] & G \times X_2 \arrow[u, "a_2"] \arrow[r, "\mathrm{id}_G\times\varphi_{21}"] & G \times X_1 \arrow[u, "a_1"] \arrow[r, "\mathrm{id}_G\times\varphi_{10}"] & G \times X_0 \arrow[u, "a_0"] \end{tikzcd}$$

[28] A topological space is *locally compact* if there is a basis of (open) subsets each having *compact closure.*

[29] A topological space is *Hausdorff* if any two distinct points have neighborhoods disjoint from each other.

[30] A topological space X has a *countable* basis if, as suggested by the terminology, it has a basis that is countable.

Then there is a unique continuous group action $a : G \times X \to X$ on the projective limit X such that we have a commutative diagram [31]

$$\begin{tikzcd}
X \arrow[rrr, bend left, "p_2"] \arrow[rrrr, bend left, "p_1"] \arrow[rrrrr, bend left, "p_0"] & & \cdots \arrow[r, "\varphi_{32}"] & X_2 \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
G \times X \arrow[u, dashed, "a"] \arrow[rrr, bend right, "\mathrm{id}_G \times p_2"'] \arrow[rrrr, bend right, "\mathrm{id}_G \times p_1"'] \arrow[rrrrr, bend right, "\mathrm{id}_G \times p_0"'] & & \cdots \arrow[r, "\mathrm{id}_G \times \varphi_{32}"] & G \times X_2 \arrow[u, "a_2"] \arrow[r, "\mathrm{id}_G \times \varphi_{21}"] & G \times X_1 \arrow[u, "a_1"] \arrow[r, "\mathrm{id}_G \times \varphi_{10}"] & G \times X_0 \arrow[u, "a_0"]
\end{tikzcd}$$

Proof: Composing the maps $\mathrm{id}_G \times p_n : G \times X \to G \times X_n$ with the action map $a_n : G \times X_n \to X_n$ gives a compatible family of maps $G \times X \to X_n$. By definition of the projective limit X, we have a unique map $G \times X \to X$ making the diagram commute, as claimed.

But we should really check the associativity property $(gh)x = g(hx)$ required of a group action, with $g, h \in G$ and $x \in X$, not to mention the condition $e_G x = x$. (Unsurprisingly, it turns out fine.) We need to rewrite the associativity in terms of maps. In a diagram, the associativity of the action on X_n asserts the commutativity of the triangle

$$\begin{tikzcd}
& X_n & \\
G \times G \times X_n \arrow[ur, "{(g,h,x) \to g(h(x))}"] \arrow[rr, equal, "\mathrm{id}_G \times \mathrm{id}_G \times \mathrm{id}_{X_n}"] & & G \times G \times X_n \arrow[ul, "{(g,h,x) \to (gh)(x)}"']
\end{tikzcd}$$

That is, associativity is equivalent to the *equality* of two maps $G \times G \times X_n \to X_n$. Thus, by the uniqueness of the induced map on the projective limit X, we obtain the same limit maps $G \times G \times X \to X$. This gives the associativity on the projective limit from the known associativities. We did not bother to prove that the identity acts trivially. ///

In a similar vein, thinking of our glib presumption that the projective limit $\mathbb{Z}_2$ of the groups $\mathbb{Z}/2^n\mathbb{Z}$ was a group, not to mention a *topological* group, we should verify these things.

[3.0.2] Claim: Projective limits of topological groups, with all but finitely many *compact*, are topological groups. [32] Further, *countable* projective limits [33] of *countably-based* topological groups have countable bases.

[3.0.3] Remark: The proof has several parts, which show somewhat more than the claim asserts. For example, it becomes clear that arbitrary projective limits of groups exist. Arbitrary projective limits of

[31] In this diagram, there is no claim that $G \times X$ is the projective limit of the objects on the bottom row, only that the maps to the top row exist as indicated.

[32] That is, such projective limits of topological groups *exist*, as topological groups.

[33] All our diagrams have implicitly used only countably-many objects in the family from which the projective limit is formed. Nevertheless, this countability is not mandated in a more general notion of projective limit, so should be explicitly noted when it matters.

Hausdorff spaces are Hausdorff. Projective limits of families of locally compact Hausdorff spaces X_i, *with all but finitely many X_i compact*, are locally compact. And countable limits of countably-based topological spaces are countably-based.

Proof: Let

$$\begin{tikzcd} G \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & G_1 \arrow[r, "\varphi_{10}"] & G_0 \end{tikzcd}$$

be a projective limit of topological groups, where each transition map $\varphi_{i,i-1}$ is a continuous group homomorphism, and the p_i are continuous maps from the projective limit object G. All that we truly know about G and the p_i at the outset is that G is a topological space and that the p_i are continuous. We must prove that G is a group, in fact a topological group, and that the p_i are group homomorphisms.

First, we need to find the very *definition* of the alleged group operation $G \times G \to G$ on the limit object, much as we defined the group action on a limiting object above. Of course, this must be some sort of limit of the multiplication maps $\mu_n : G_n \times G_n \to G_n$ by $\mu_n : g \times h \to gh$. At the same time, to make a map $G \times G \to G$ is to make a compatible family of maps $f_n : G \times G \to G_n$. Indeed, let

$$f_n = \mu_n \circ (p_n \times p_n) : G \times G \to G_n$$

That is, there is a unique $\mu : G \times G \to G$ induced by the f_n making a commutative diagram

$$\begin{tikzcd} G \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & G_1 \arrow[r, "\varphi_{10}"] & G_0 \\ G \times G \arrow[u, dotted, "\mu"] \arrow[urr, dashed, "f_1"] \arrow[urrr, dashed, "f_0"] \arrow[rr, bend right, "p_1 \times p_1"'] \arrow[rrr, bend right, "p_0 \times p_0"'] & \cdots \arrow[r, "\varphi_{21} \times \varphi_{21}"] & G_1 \times G_1 \arrow[u, "\mu_1"] \arrow[r, "\varphi_{10} \times \varphi_{10}"] & G_0 \times G_0 \arrow[u, "\mu_0"'] \end{tikzcd}$$

The associativity $a(bc) = (ab)c$ of the alleged[34] group operation comes (much as in the discussion of group actions on limits), first from the commutativity of the diagrams

$$\begin{tikzcd} & G_n & \\ G_n \times G_n \times G_n \arrow[ur, "(a,b,c) \to a(bc)"] \arrow[rr, equal, "\mathrm{id}_{G_n} \times \mathrm{id}_{G_n} \times \mathrm{id}_{G_n}"] & & G_n \times G_n \times G_n \arrow[ul, "(a,b,c) \to (ab)c"'] \\ G \times G \times G \arrow[u, "p_n \times p_n \times p_n"] \arrow[rr, equal, "\mathrm{id}_G \times \mathrm{id}_G \times \mathrm{id}_G"] & & G \times G \times G \arrow[u, "p_n \times p_n \times p_n"'] \end{tikzcd}$$

which proves that the two different maps $G \times G \times G \to G_n$ are the same, and, second, the uniqueness of the

[34] In fact, it is slightly dangerous to use this notation, since it makes it too easy to lose track of what we truly know, versus what must be shown. The associativity we want to prove is properly written as $\mu(a, \mu(b, c)) = \mu(\mu(a, b), c))$.

dotted induced map in

$$\begin{tikzcd}
G \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & G_1 \arrow[r, "\varphi_{10}"] & G_0 \\
G\times G\times G \arrow[u, dotted] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[rr, bend right, "p_1\times p_1\times p_1"] \arrow[rrr, bend right, "p_0\times p_0\times p_0"'] & \cdots \arrow[r, "\varphi_{21}\times\varphi_{21}\times\varphi_{21}"] & G_1\times G_1\times G_1 \arrow[u, "a(bc)=(ab)c"'] \arrow[r, "\varphi_{10}\times\varphi_{10}\times\varphi_{10}"] & G_0\times G_0\times G_0 \arrow[u, "a(bc)=(ab)c"']
\end{tikzcd}$$

The identity element e in the limit is specified as a sort of limit of the identities e_n in G_n, specifically, as the image $f(s)$ of the induced map f in the diagram

$$\begin{tikzcd}
G \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & G_1 \arrow[r, "\varphi_{10}"] & G_0 \\
& \{s\} \arrow[ul, dashed, "f"] \arrow[ur, "s\to e_1"] \arrow[urr, "s\to e_0"'] & &
\end{tikzcd}$$

Existence of an inversion map (and its property) is a further exercise in this technique, which we leave to the reader. Thus, the map $\mu : G\times G \to G$ does have the properties of a group operation on G.

To show that the projections $p_n : G \to G_n$ are group homomorphisms, we note that to say that $f : A \to B$ is a group homomorphism for groups A, B is to require the commutativity of the square

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
A\times A \arrow[u, "\mu_A"] \arrow[r, "f\times f"] & B\times B \arrow[u, "\mu_B"']
\end{tikzcd}$$

where μ_A and μ_B are the multiplication maps belonging to A, B, respectively. In the case at hand, we would *want* the commutativity of

$$\begin{tikzcd}
G \arrow[r, "p_n"] & G_n \\
G\times G \arrow[u, "\mu"] \arrow[r, "p_n\times p_n"] & G_n\times G_n \arrow[u, "\mu_n"']
\end{tikzcd}$$

Happily, the commutativity of these squares is part of the commutativity of the diagram *defining* the multiplication $\mu : G\times G \to G$. That is, the fact that the projections are group homomorphisms is a by-product of the construction of the multiplication on G.

The Hausdorff-ness of the limit will follow from the earlier observation that a limit $\lim_i X_i$ is a subspace of the corresponding product $\Pi_i X_i$. An arbitrary product of Hausdorff spaces is Hausdorff. [35] And arbitrary

[35] That products of Hausdorff spaces are Hausdorff has a natural proof, as follows. Given $x \neq y$ in the product of spaces X_i, there is at least one index j such that the projection p_j of the product to X_j distinguishes x and y, that is, such that $p_j x \neq p_j y$. (This assertion itself can be proven as an exercise using the mapping property definition of *product*, as suggested in an earlier handout.) Since X_j is Hausdorff, there are disjoint neighborhoods U_j and V_j of $p_j x$ and $p_j y$. Perhaps using the explicit construction of products as *cartesian* products, let $U = U_j \times \Pi_{i\neq j} X_j$ and $V = V_j \times \Pi_{i\neq j} X_j$. These are disjoint neighborhoods of x and y.

subspaces of Hausdorff spaces, given the subspace topology, are Hausdorff. [36] Thus, limits of Hausdorff spaces are Hausdorff.

Similarly, the local compactness of limits of locally compact topological spaces X_i, with all but finitely many compact, will follow from the analogous assertion for *products*. First, observe that a basic open $\Pi_i U_i$ in a product has *closure* the product of the closures $\overline{U_i}$ of the factors U_i. [37] When all the $\overline{U_i}$ are compact, the product is compact, by Tychonoff's theorem. Since only *finitely-many* X_i are not in fact compact themselves, but in any case are still *locally* compact, inside such factors the product topology allows us to take compact-closure neighborhoods of any point. Thus, every point in the product has a neighborhood (in fact, a *basic* neighborhood) with compact closure.

Finally, to discuss countable-based-ness, it suffices to prove that countable *products* of countably-based topological spaces X_i are countably-based, since limits are subspaces of products. A basis for a product topology consists of products $\Pi_i U_i$ where for each i the set U_i is in a countable basis for X_i, and for all but finitely-many indices i the set U_i is just X_i. To count these possibilities, note first that there are only countably-many *finite* subsets of a countable set. Next, for each finite subset $\{i_1, \ldots, i_n\}$ of the countable indexing set, there are only countably-many choices of

$$U_{i_1}, \ \ldots, \ U_{i_n} \quad (\text{with } U_{i_j} \text{ a basis element in } X_{i_j})$$

The sum of countably many countables is countable. ///

[3.0.4] Remark: The *box* topology behaves worse than the genuine product topology with regard to preservation of countable-based-ness: since there are *uncountably* many *not-necessarily-finite* subsets of a countable index set, a product of (infinitely) countably-many countably-based spaces, with the *box* topology, will *not* be countably-based.

Having verified that things work as hoped, especially that topological aspects and group-theoretic aspects are captured, we return to the solenoid.

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4. *Automorphisms of solenoids, again*

Having cleaned up our viewpoint, we can give an economical and rigorous treatment of the automorphisms found earlier of the solenoid

$$\begin{tikzcd} X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\text{mod } 2"] & \mathbb{R}/2\mathbb{Z} \arrow[r, "\text{mod } 1"] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

Our aim is to prove that we have a transitive (continuous) group action of $\mathbb{R} \times \mathbb{Z}_2$, with an isotropy group a diagonal copy $\mathbb{Z}^\Delta$ of the integers $\mathbb{Z}$, and, thus, that

$$2\text{-solenoid} \approx (\mathbb{R} \times \mathbb{Z}_2)/\mathbb{Z}^\Delta$$

as G-spaces. [38]

---

[36] That subspaces Y of Hausdorff spaces X are Hausdorff is straightforward: given $x \neq y$ in Y, let U, V be disjoint neighborhoods of x, y in X. Then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x, y in Y.

[37] The product of the closures $\overline{U_i}$ of the opens U_i is a closed set (from the definition of product topology) and *contains* the product of the U_i. On the other hand, let x be in the product of the closures. Then every basic neighborhood $\Pi_i V_i$ of x (with V_i open in X_i) has the property that V_i meets U_i, since $p_i x$ is in the closure of U_i. That is, $\Pi_i V_i$ meets $\Pi_i U_i$, so x is in the closure of the product. This proves the equality.

[38] Again, the notion of G-space is the reasonable one, of topological spaces acted upon continuously by a topological group G. A map of G-spaces $\psi : A \to B$ is a continuous map of topological spaces which respects the action of G, in the sense that $\psi(g \cdot a) = g \cdot \psi(a)$.

First, we have an induced continuous group action $\mathbb{R} \times X \to X$ (the dotted arrow below) induced by the compatible family of (dashed arrow) maps $\mathbb{R} \times X \to \mathbb{R}/2^n\mathbb{Z}$ created by composition of the actions $\mathbb{R} \times \mathbb{R}/2^n\mathbb{Z} \to \mathbb{R}/2^n\mathbb{Z}$ with the projection $\mathbb{R} \times X \to \mathbb{R} \times \mathbb{R}/2^n\mathbb{Z}$, in

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & \mathbb{R}/2\mathbb{Z} \arrow[r, "\varphi_{10}"] & \mathbb{R}/\mathbb{Z} \\
\mathbb{R} \times X \arrow[u, dotted] \arrow[urr, dashed] \arrow[urrr, dashed] \arrow[rr, bend right, "\mathrm{id}_{\mathbb{R}} \times p_1"'] \arrow[rrr, bend right, "\mathrm{id}_{\mathbb{R}} \times p_0"'] & \cdots \arrow[r, "\mathrm{id}_{\mathbb{R}} \times \varphi_{21}"] & \mathbb{R} \times \mathbb{R}/2\mathbb{Z} \arrow[u, "(r,x_1)\to r+x_1"'] \arrow[r, "\mathrm{id}_{\mathbb{R}} \times \varphi_{10}"] & \mathbb{R} \times \mathbb{R}/\mathbb{Z} \arrow[u, "(r,x_1)\to r+x_0"']
\end{tikzcd}$$

The diagrammatic form of the action of the projective limit (countably-based topological) group

$$\begin{tikzcd}
\mathbb{Z}_2 \arrow[rr, bend left, "\text{mod } 2"] \arrow[rrr, bend left, "\text{mod } 1"] & \cdots \arrow[r, "\text{mod } 2"] & \mathbb{Z}/2^1\mathbb{Z} \arrow[r, "\text{mod } 1"] & \mathbb{Z}/2^0\mathbb{Z}
\end{tikzcd}$$

on the solenoid X is nearly identical, with the minor complication that the action of $\mathbb{Z}_2$ on $\mathbb{R}/2^n\mathbb{Z}$ is via the image group $\mathbb{Z}/2^n\mathbb{Z}$ action on $\mathbb{R}/2^n\mathbb{Z}$, *by definition.*

Next, we want to prove *transitivity* of the joint action $\mathbb{R} \times \mathbb{Z}_2$ on the solenoid. Specify a point x on the solenoid by a compatible family of maps (and the induced map f to X)

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\text{mod } 2"] & \mathbb{R}/2\mathbb{Z} \arrow[r, "\text{mod } 1"] & \mathbb{R}/\mathbb{Z} \\
& \{s\} \arrow[ul, dashed, "f"] \arrow[ur, "f_1"] \arrow[urr, "f_0"'] & &
\end{tikzcd}$$

The action of $\mathbb{R}$ is transitive on the rightmost circle $\mathbb{R}/\mathbb{Z}$, so is transitive on maps f_0 from $\{s\}$ to that circle. Thus, given a point f on the solenoid (given by a family $\{f_n\}$ of maps from $\{s\}$), we adjust it by $\mathbb{R}$ so that $f_0(s) = 0$ in $\mathbb{R}/\mathbb{Z}$.

Then the compatibility condition on the images $f_n(s)$ requires that, given $f_0(s) = 0$, all $f_n(s)$ are inside $\mathbb{Z}/2^n\mathbb{Z} \subset \mathbb{R}/2^n\mathbb{Z}$. That is, the family of maps f_n gives a compatible family

$$\begin{tikzcd}
{} & \cdots \arrow[r, "\text{mod } 2"] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "\text{mod } 1"] & \mathbb{Z}/\mathbb{Z} \\
& \{s\} \arrow[ul, dashed] \arrow[ur, "f_1"] \arrow[urr, "f_0"'] & &
\end{tikzcd}$$

which is exactly our definition of $\mathbb{Z}_2$. Thus, visibly this $\mathbb{Z}_2$ maps all these points to 0. This proves the transitivity.

Thus, certainly as *sets*,

$$\text{2-solenoid} \approx (\mathbb{R} \times \mathbb{Z}_2)/\mathbb{Z}^\Delta$$

The surprising result proved in the appendix will imply that this is a *topological* isomorphism, if we are sure that $\mathbb{R} \times \mathbb{Z}_2$ has a countable basis. It is standard [39] that $\mathbb{R}$ has a countable basis. It is less standard, but still standard in light of our earlier discussion, that $\mathbb{Z}_2$ has a countable basis, since it is a countable projective limit of countably-based spaces. [40]

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5. *Appendix: uniqueness of projective limits*

As an exercise in proving the uniqueness-up-to-unique-isomorphism (assuming *existence*) of things specified by universal mapping properties, we carry out the proof of uniqueness of projective limits. Part of the point of the exercise is reiteration of the inessentialness of the details of the situation. In particular, as above, a mapping-property approach provides a very useful packaging for topological details that might otherwise be burdensome.

Thus, given topological spaces X_i with continuous maps

$$\cdots \xrightarrow{\varphi_{21}} X_1 \xrightarrow{\varphi_{10}} X_0$$

let X and projections p_i and Y and projections q_i fit into diagrams

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[rr, bend left, "q_1"] \arrow[rrr, bend left, "q_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0
\end{tikzcd}$$

such that, for all families of maps $f_i : Z \to X_i$ such that all triangles commute in

$$\begin{tikzcd}
\cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
\cdots & & \\
Z \arrow[uur, "f_1"] \arrow[uurr, "f_0"'] & &
\end{tikzcd}$$

there are unique maps $f : Z \to X$ and $g : Z \to Y$ such that all triangles commute in both diagrams

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
& \cdots & & \\
& Z \arrow[uul, dashed, "f"] \arrow[uur, "f_1"] \arrow[uurr, "f_0"'] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[rr, bend left, "q_1"] \arrow[rrr, bend left, "q_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
& \cdots & & \\
& Z \arrow[uul, dashed, "g"] \arrow[uur, "f_1"] \arrow[uurr, "f_0"'] & &
\end{tikzcd}$$

Then

---

[39] The space $\mathbb{R}$ has a countable basis consisting of open balls with rational radii centered at rational points.

[40] Again, it is easy to see that a countable product of countably-based spaces is countably-based, and the projective limit can be realized as a *subspace* of the product, so is countably-based.

[5.0.1] Claim: There is a unique isomorphism $q : Y \to X$ such that we have a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] \arrow[d, equal] & X_0 \arrow[d, equal] \\
Y \arrow[u, "q"] \arrow[rr, bend right, "q_1"'] \arrow[rrr, bend right, "q_0"'] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0
\end{tikzcd}$$

Proof: First, we prove that the only map of a projective limit to itself compatible with all projections is the identity map. That is, using $p_i : X \to X_i$ itself in the role of $f_i : Z \to X_i$, we find a unique map $p : X \to X$ such that all triangles commute in

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
& X \arrow[ul, dashed, "p"] \arrow[u, phantom, "\cdots"] \arrow[ur, "p_1"] \arrow[urr, "p_0"'] & &
\end{tikzcd}$$

Since the identity map id_X fits the role of p, by uniqueness p can *only* be the identity on X.

Now we can do the main part of the proof. Let $q_i : Y \to X_i$ take the role of $f_i : Z \to X_i$. Then there is a unique $q : Y \to X$ such that all triangles commute in

$$\begin{tikzcd}
X \arrow[rr, bend left, "p_1"] \arrow[rrr, bend left, "p_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
& Y \arrow[ul, dashed, "q"] \arrow[u, phantom, "\cdots"] \arrow[ur, "q_1"] \arrow[urr, "q_0"'] & &
\end{tikzcd}$$

To show that q is an isomorphism, reverse the roles of X and Y. Then there is a unique $p : X \to Y$ such that all triangles commute in

$$\begin{tikzcd}
Y \arrow[rr, bend left, "q_1"] \arrow[rrr, bend left, "q_0"] & \cdots \arrow[r, "\varphi_{21}"] & X_1 \arrow[r, "\varphi_{10}"] & X_0 \\
& X \arrow[ul, dashed, "p"] \arrow[u, phantom, "\cdots"] \arrow[ur, "p_1"] \arrow[urr, "p_0"'] & &
\end{tikzcd}$$

Then $p \circ q : Y \to Y$ and $q \circ p : X \to X$ are maps compatible with projections, so must be the identities, by the first point of this argument. That is, these are mutually inverse maps, so q is an isomorphism. ///

[5.0.2] **Remark:** As usual in these categorical arguments, any *continuity* or other requirements on the maps are packaged (or hidden) in the quantification over all families of maps $f_i : Z \to X_i$. That is, the *implicit* specification that Z be a topological space and f_i be continuous are what make the result relevant to topological spaces and continuous maps. Thus, despite the lack of overt references to topology, the uniqueness proven above yields *topological* isomorphisms, not merely *set* isomorphisms.

[5.0.3] **Remark:** As in our earlier discussion of the point that a projective limit of *groups* is a *group*, the additional structure that must be demonstrated to have a group, as opposed to merely a *set*, is hidden in the proof of *existence* of a projective limit. That is, in any case there is at most one, regardless of details, but proof of existence invariable requires somewhat greater detail.

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6. *Appendix: topology of* $X \approx G/G_x$

The point of this appendix is to prove that, with mild hypotheses, a topological space X acted upon transitively by a *topological* group G is homeomorphic to the quotient G/G_x, where G_x is the isotropy group of a chosen point x in X.

By the way, since we are *not* wanting to assume a pre-existing mastery of point-set topology, much less a mastery of ideas about topological *groups*, several basic ideas will need to be developed in the course of the proof. Everything here is completely standard and widely useful. The discussion includes a form of the *Baire Category Theorem* [41] for locally compact Hausdorff spaces.

[6.0.1] **Remark:** Ignoring the topology, that is, as *sets*, the bijection $G/G_x \approx X$ is easy to see, and the proof needs nothing. The *topological* aspects are not trivial, by contrast, and it should come as a surprise that the topology of the group G completely determines the topology of the set X on which it acts.

[6.0.2] **Proposition:** Let G be a locally compact, Hausdorff topological group[42] and X a locally compact Hausdorff topological space with a continuous transitive action of G upon X. [43] Suppose that G has a *countable basis.* [44] Let x be any fixed element of X, and G_x the isotropy group[45] The natural map

$$G/G_x \to X \text{ by } gG_x \to gx$$

is a homeomorphism.

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[41] The more common form of the Baire Category Theorem asserts that a *complete metric space* is *not* a countable union of closed sets each containing no non-empty open set.

[42] As expected, this means that G is a group and is a topological space, the group multiplication is a continuous map $G \times G \to G$, and inversion is continuous. The *local compactness* is the requirement that every point has an open neighborhood with compact closure. The Hausdorff requirement is that any two distinct points $x \neq y$ have open neighborhoods $U \ni x$ and $V \ni y$ that are disjoint, that is, $U \cap V = \phi$.

[43] As expected, continuity of the action means that $G \times X \to X$ by $g \times x \to gx$ is continuous. The transitivity means that for any $x \in X$ the set of images of x by elements of G is the whole set X, that is, $\{gx : g \in G\} = X$.

[44] That is, there is a countable collection B (the basis) of open sets in G such that *any* open set is a union of sets from the basis B.

[45] As usual, the isotropy (sub-) group of x in G is the subgroup of group elements fixing x, namely, $G_x := \{g \in G : gx = x\}$.

Proof: We must do a little systematic development of the topology of topological groups in order to give a coherent argument.

[6.0.3] Claim: In a locally compact Hausdorff space X, given an open neighborhood U of a point x, there is a neighborhood V of x with compact closure $\overline{V}$ and $\overline{V} \subset U$.

Proof: By local compactness, x has a neighborhood W with compact closure. Intersect U with W if necessary so that U has compact closure $\overline{U}$. Note that the compactness of $\overline{U}$ implies that the *boundary* [46] ∂U of U is compact. Using the Hausdorff-ness, for each $y \in \partial U$ let W_y be an open neighborhood of y and V_y an open neighborhood of x such that $W_y \cap V_y = \phi$. By compactness of ∂U, there is a finite list $y_1, \ldots, y_n$ of points on ∂U such that the sets U_{y_i} cover ∂U. Then $V = \bigcap_i V_{y_i}$ is open and contains x. Its closure is contained in $\overline{U}$ and in the complement of the open set $\bigcup_i W_{y_i}$, the latter containing ∂U. Thus, the closure $\overline{V}$ of V is contained in U. ///

[6.0.4] Claim: The map $gG_x \to gx$ is a continuous bijection of G/G_x to X.

Proof: First, $G \times X \to X$ by $g \times y \to gy$ is continuous by definition of the continuity of the action. Thus, with fixed $x \in X$, the restriction to $G \times \{x\} \to X$ is still continuous, so $G \to X$ by $g \to gx$ is continuous. The quotient topology on G/G_x is the unique topology on the *set* (of cosets) G/G_x such that any continuous $G \to Z$ constant on G_x cosets factors through the quotient map $G \to G/G_x$. That is, we have a commutative diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & Z \\ G/G_x \arrow[ur, dashed] & \end{tikzcd}$$

Thus, the induced map $G/G_x \to X$ by $gG_x \to gx$ is continuous. ///

[6.0.5] Remark: We need to show that $gG_x \to gx$ is *open* to prove that it is a homeomorphism.

[6.0.6] Claim: For a given point $g \in G$, every neighborhood of g is of the form gV for some neighborhood V of 1.

Proof: First, again, $G \times G \to G$ by $g \times g \to gh$ is continuous, by assumption. Then, for fixed $g \in G$, the map $h \to gh$ is continuous on G, by restriction. And this map has a continuous inverse $h \to g^{-1}h$. Thus, $h \to gh$ is a homeomorphism of G to itself. In particular, since $1 \to g \cdot 1 = g$, neighborhoods of 1 are carried to neighborhoods of g, as claimed. ///

[6.0.7] Claim: Given an open neighborhood U of 1 in G, there is an open neighborhood V of 1 such that $V^2 \subset U$, where

$$V^2 = \{gh : g, h \in V\}$$

Proof: The continuity of $G \times G \to G$ assures that, given the neighborhood U of 1, the inverse image W of U under the multiplication $G \times G \to G$ is open. Since $G \times G$ has the product topology, W contains an open of the form $V_1 \times V_2$ for opens V_i containing 1. With $V = V_1 \cap V_2$, we have $V^2 \subset V_1 \cdot V_2 \subset U$ as desired.
///

[6.0.8] Remark: Similarly, but more simply, since inversion $g \to g^{-1}$ is continuous and is its own (continuous) inverse, for an open set V the image $V^{-1} = \{g^{-1} : g \in V\}$ is open. Thus, for example,

[46] As usual, the *boundary* of a set E in a topological space is the intersection $\overline{E} \cap \overline{E^c}$ of the closure $\overline{E}$ of E and the closure $\overline{E^c}$ of the complement E^c of E.

given a neighborhood V of 1, replacing V by $V \cap V^{-1}$ replaces V by a smaller *symmetric* neighborhood, meaning that the new V satisfies $V^{-1} = V$.

The following result is not strictly necessary, but sheds some light on the nature of topological groups.

[6.0.9] Claim: Given a set E in G,

$$\text{closure } E = \bigcap_U E \cdot U$$

where U runs over open neighborhoods of 1. [47]

Proof: A point $g \in G$ is in the closure of E if and only if every neighborhood of g meets E. That is, from just above, every set gU meets E, for U an open neighborhood of 1. That is, $g \in E \cdot U^{-1}$ for every neighborhood U of 1. We have noted that inversion is a homeomorphism of G to itself (and sends 1 to 1), so the map $U \to U^{-1}$ is a bijection of the collection of neighborhoods of 1 to itself. Thus, g is in the closure of E if and only if $g \in E \cdot U$ for every open neighborhood U of 1, as claimed. ///

[6.0.10] Remark: This allows us to give another proof, for topological groups, of the fact that, given a neighborhood U of 1 in G, there is a neighborhood V of 1 such that $\overline{V} \subset U$. (We did prove this above for locally compact Hausdorff spaces generally.)

Proof: First, from the continuity of $G \times G \to G$, there is V such that $V \cdot V \subset U$. From the previous claim, $\overline{V} \subset V \cdot V$, so $\overline{V} \subset V \cdot V \subset U$, as claimed. ///

[6.0.11] Remark: We can improve the conclusion of the previous remark using the local compactness of G, as follows. Given a neighborhood U of 1 in G, there is a neighborhood V of 1 such that $\overline{V} \subset U$ and $\overline{V}$ is *compact*. Indeed, local compactness means exactly that there is a local basis at 1 consisting of opens with compact closures. Thus, given V as in the previous remark, shrink V if necessary to have the compact closure property, and still $\overline{V} \subset V \cdot V \subset U$, as claimed.

[6.0.12] Corollary: For an open subset U of G, given $g \in U$, there is a compact neighborhood V of $1 \in G$ such that $gV^2 \subset U$.

Proof: The set $g^{-1}U$ is an open containing 1, so there is an open $W \ni 1$ such that $W^2 \subset g^{-1}U$. Using the previous claim and remark, there is a compact neighborhood V of 1 such that $V \subset W$. Then $V^2 \subset W^2 \subset g^{-1}U$, so $gV^2 \subset U$ as desired. ///

[6.0.13] Claim: Given an open neighborhood V of 1, there is a countable list $g_1, g_2, \ldots$ of elements of G such that $G = \bigcup_i g_i V$.

Proof: To see this, first let $U_1, U_2, \ldots$ be a countable basis. For $g \in G$, by definition of a basis,

$$gV = \bigcup_{U_i \subset gV} U_i$$

Thus, for each $g \in G$, there is an index $j(g)$ such that $g \in U_{j(g)} \subset gV$. Do note that there are only countably many such indices. For each index i appearing as $j(g)$, let g_i be an element of G such that $j(g_i) = i$, that is,

$$g_i \in U_{j(g_i)} \subset g_i \cdot V$$

[47] This characterization of the closure of a subset of a topological group is very different from anything that happens in general topological spaces. To find a related result we must look at more restricted classes of spaces, such as *metric* spaces. In a metric space X, the closure of a set E is the collection of all points $x \in X$ such that, for every $\varepsilon > 0$, the point x is within ε of some point of E.

Then, for every $g \in G$ there is an index i such that

$$g \in U_{j(g)} = U_{j(g_i)} \subset g_i \cdot V$$

This shows that the union of these $g_i \cdot V$ is all of G. ///

A subset E of a topological space is **nowhere dense** if its closure contains no (non-empty) open set. [48]

[6.0.14] Claim: *(Variant of Baire Category theorem)* A locally compact Hausdorff topological space is *not* a countable union of nowhere dense sets. [49]

Proof: Let W_n be closed sets containing no non-empty open subsets. Thus, any non-empty open U meets the complement of W_n, and $U - W_n$ is a *non-empty* open. Let U_1 be a non-empty open with compact closure, so $U_1 - W_1$ is *non-empty* open. From the discussion above, there is a non-empty open U_2 whose *closure* is contained in $U_1 - W_1$. Continuing inductively, there are non-empty open sets U_n with compact closures such that

$$U_{n-1} - W_{n-1} \supset \bar{U}_n$$

Certainly

$$\bar{U}_1 \supset \bar{U}_2 \supset \bar{U}_3 \supset \ldots$$

Then $\bigcap \bar{U}_i \neq \phi$, by compactness. [50] [51] Yet this intersection fails to meet any W_n. In particular, it *cannot* be that the union of the W_n's is the whole space. ///

Now we can prove that $G/G_x \approx X$, using the viewpoint we've set up.

Given an open set U in G and $g \in U$, let V be a compact neighborhood of 1 such that $gV^2 \subset U$. Let $g_1, g_2, \ldots$ be a countable set of points such that $G = \bigcup_i g_i V$. Let $W_n = g_n V x \subset X$. By the transitivity, $X = \bigcup_i W_i$.

We observed at the beginning of this discussion that $G \to X$ by $g \to gx$ is continuous, so W_n is compact, being a continuous image of the compact set $g_n V$. So W_n is closed since it is a compact subset of the Hausdorff space X.

By the (variant) Baire category theorem, some $W_m = g_m V x$ contains a non-empty open set S of X. For $h \in V$ so that $g_m h x \in S$,

$$gx = g(g_m h)^{-1}(g_m h)x \in g h^{-1} g_m^{-1} S$$

Every group element $y \in G$ acts by homeomorphisms of X to itself, since the continuous inverse is given by y^{-1}. Thus, the image $gh^{-1}g_m^{-1}S$ of the open set S is open in X. Continuing,

$$gh^{-1}g_m^{-1}S \subset gh^{-1}g_m^{-1}g_m V x \subset gh^{-1}Vx \subset gV^{-1} \cdot Vx \subset Ux$$

---

[48] The union of all open subsets of a given set is its *interior*. Thus, a set is nowhere dense if its closure has empty interior.

[49] The more common verison of the Baire category theorem asserts the same conclusion for *complete metric* spaces. The argument is structurally identical.

[50] In Hausdorff topological spaces X compact sets C are closed, proven as follows. Fixing x not in C, for each $y \in C$, there are opens $U_y \ni x$ and $V_y \ni y$ with $U \cap V = \phi$, by the Hausdorff-ness. The U_y's cover C, so there is a finite subcover, $U_{y_1}, \ldots, U_{y_n}$, by compactness. The finite *intersection* $W_x = \bigcap_i V_{y_i}$ is open, contains x, and is disjoint from C. The union of all W_x's for $x \notin C$ is open, and is exactly the complement of C, so C is closed.

[51] The intersection of a nested sequence $C_1 \supset C_2 \supset \ldots$ of non-empty compact sets C_n in a Hausdorff space X is non-empty. Indeed, the complements $C_n^c = X - C_n$ are open (since compact sets are closed in Hausdorff spaces), and if the intersection were empty, then the union of the opens C_n^c would cover C_1. By compactness of C_1, there is a finite subcollection $C_1^c, \ldots, C_n^c$ covering C_1. But $C_1^c \subset \ldots \subset C_n^c$, and C_n^c omits points in C_n, which is non-empty, contradiction.

Therefore, gx is an interior point of Ux, for all $g \in U$. ///