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Basic complex analysis

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Complex analysis is one of the most natural and productive continuations of basic calculus, *not* addressing pathologies and pitfalls, but, instead, showing that natural, well-behaved functions are even better than imagined.

Mercifully, very many functions arising in practice are indeed natural and well-behaved in the relevant sense, so avoid pathologies, and behave even better than we had hoped. Arguably, the first 150 years of calculus in fact addressed such functions, thereby discovering the further remarkable usefulness of calculus-as-complex-analysis long before anyone thought to worry about the subtler distinctions and troubles highlighted in the 19th century. Thus, arguably, Euler, Lagrange, and the most effective of their contemporaries, to some degree inadvertently thought in terms we can now reinterpret as justifiable as *complex analysis*.

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1. *Complex differentiation*

[1.1] Complex differentiation For complex-valued f on an open set $\Omega \subset \mathbb{C}$, the *complex derivative* $f'(z)$, *if it exists*, is

$$f'(z) \;=\; \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \qquad\qquad \text{(for complex } h \to 0\text{)}$$

It is critical that the limit exist for *complex* h approaching 0. If the limit exists for all $z \in \Omega$, say f is *complex differentiable* on Ω.

Given a function f on a region Ω, when there is a complex-differentiable F with $F' = f$, say that F is a *primitive* of f on Ω. This is obviously a kind of *anti-derivative*.

[1.2] Examples *Polynomial* functions in z are complex-differentiable, with the same differentiation formulas as in single-real-variable calculus, because those results are essentially algebraic:

$$(z^n)' \;=\; \lim_{h\to 0} \frac{(z+h)^n - z^n}{h} \;=\; \lim_{h\to 0} \frac{z^n + nhz^{n-1} + h^2(\ldots) - z^n}{h}$$

$$=\; \lim_{h\to 0} \frac{nhz^{n-1} + h^2(\ldots)}{h} \;=\; \lim_{h\to 0} \Big(nz^{n-1} + h(\ldots)\Big) \;=\; nz^{n-1}$$

[1.3] $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$ and Cauchy-Riemann equation From the notation, and as applied to polynomials in z, it seems that complex differentiation is application of $\frac{\partial}{\partial z}$. What about $\overline{z}$? For the moment, with $z = x + iy$, we simply *declare*

$$\frac{\partial}{\partial z} = \tfrac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)$$

The signs are explained/remembered by checking that

$$\frac{\partial}{\partial z} z^n = \tfrac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)(x+iy)^n = \tfrac{1}{2}\Big(n(x+iy)^{n-1} - i\cdot i\cdot n(x+iy)^{n-1}\Big)$$

$$= \tfrac{1}{2}\Big(n(x+iy)^{n-1} + n(x+iy)^{n-1}\Big) = nz^{n-1}$$

Thus, yes, there is the perhaps-surprising outcome

$$\frac{\partial}{\partial \overline{z}} z^n = \tfrac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)(x+iy)^n = \tfrac{1}{2}\Big(n(x+iy)^{n-1} + i\cdot i\cdot n(x+iy)^{n-1}\Big)$$

$$= \tfrac{1}{2}\Big(n(x+iy)^{n-1} - n(x+iy)^{n-1}\Big) = 0$$

The latter is correct, despite appearing to be in conflict with the idea that *knowing* z or $\overline{z}$ determines the other, but a partial derivative of one thing with respect to another being 0 means they're *independent*.

Thus, complex-differentiable f satisfies

$$\frac{\partial}{\partial \overline{z}} f(z) = \tfrac{1}{2}\Big(\lim_{\text{real } \delta\to 0} \frac{f(x+\delta+iy) - f(x+iy)}{\delta} + i\frac{f(x+i(y+\delta)) - f(x+iy)}{\delta}\Big)$$

$$= \lim_{\text{real } \delta\to 0} \frac{f(x+\delta+iy) - f(x+iy)}{\delta} - \frac{f(x+iy+i\delta) - f(x+iy)}{i\delta}\Big) = \tfrac{1}{2}\big(f'(z) - f'(z)\big) = 0$$

That is, complex-differentiable f satisfies

$$\frac{\partial}{\partial \overline{z}} f(z) = 0 \qquad \text{(Cauchy-Riemann equation)}$$

This can be written in various equivalent forms, refering to real and imaginary parts separately, and/or writing out the apparent definition of $\frac{\partial}{\partial \overline{z}}$.

[1.3.1] Remark: The converse is also true: a nice-enough function satisfying the Cauchy-Riemann equation is complex-differentiable.

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2. *Exponentials, trigonometric functions*

[2.1] The exponential function The exponential function's power series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

arises from the idea that $b^{x+y} = b^x \cdot b^y$ for any $b > 0$ and $x, y \in \mathbb{R}$. [1] Thus,

$$\frac{d}{dx} b^x = \lim_{h\to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h\to 0} \frac{b^x \cdot b^h - b^x}{h} = b^x \cdot \lim_{h\to 0} \frac{b^h - 1}{h}$$

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[1] As usual, b^n is first defined for positive integers n as repeated multiplication, then for negative integer $-n$ by $b^{-n} = 1/b^n$, then for rationals via $b^{1/n} = \sqrt[n]{b}$, and then for real exponents by taking limits of rational exponents.

because multiplication is *continuous*. *Presuming* that the limit *exists*, a constant depending on $b > 0$, we'd like the simplest outcome, namely, that this constant limit is 1, and find out a little later that $b = e = 2.71828\ldots$ in that case. *Presume* that the differential equation

$$\frac{d}{dx} f(x) \;=\; f(x)$$

with $f(0) = 1$ has a convergent power series expansion $f(x) = 1 + c_1 x + c_2 x^2 + \ldots$, solve iteratively for c_n's by differentiating term-wise [2] and equate coefficients

$$c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots \;=\; 1 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

giving the pattern $n \cdot c_n = c_{n-1}$, so $c_n = 1/n!$, giving the power series for e^x. Since the factorial $n!$ grows faster than any power x^n, for all x, the power series for e^x converges absolutely for all *real* x, and, similarly, for all *complex* x. [3]

To see what number the base e is, use $e^1 = e$:

$$e \;=\; 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots \;=\; 2.71828\ldots$$

We can *check* that this has achieved the desired effect:

[2.1.1] Claim:

$$e^{z+w} \;=\; e^z \cdot e^w \qquad\qquad (\text{for } z, w \in \mathbb{C})$$

[2.1.2] Remark: For non-real complex z, there is no need to try to define e^z as a limit of simpler things, apart from the value of the power series as a limit of its finite partial sums.

Proof: This will follow from the *binomial theorem*

$$(x+y)^n \;=\; x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-2} x^2 y^{n-2} + \binom{n}{n-1} x y^{n-1} + y^n$$

with the usual binomial coefficients $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$. Compute directly

$$e^{z+w} \;=\; \sum_{n \ge 0} \frac{(z+w)^n}{n!} \;=\; \sum_{n,k} \binom{n}{k} \frac{z^{n-k} w^k}{n!} \;=\; \sum_{n,i} \frac{n!}{k!\,(n-k)!} \frac{z^{n-k} w^k}{n!}$$

The $n!$'s cancel. Letting $\ell = n - k$, this gives

$$e^{z+w} \;=\; \sum_{\ell,k} \frac{1}{k!\,\ell!} z^\ell w^k \;=\; \Big(\sum_\ell \frac{z^\ell}{\ell!}\Big)\Big(\sum_k \frac{w^k}{k!}\Big) \;=\; e^z\, e^w$$

as desired. ///

[2.1.3] Corollary: The complex conjugate $\overline{e^z}$ of e^z is $\overline{e^z} = e^{\overline{z}}$, and $|e^{ix}| = 1$ for *real* x.

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[2] To know that a convergent power series really *can* be differentiated term-wise is believable, but not completely trivial to prove: this is *Abel's theorem* below.

[3] The convergence of the power series for e^x is also *uniform* on *compact* (closed and bounded) subsets of $\mathbb{R}$ or of $\mathbb{C}$.

Proof: Since complex conjugation is a *continuous* map from $\mathbb{C}$ to itself, respecting addition and multiplication,

$$\overline{e^z} \;=\; \overline{1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots} \;=\; 1 + \frac{\overline{z}}{1} + \frac{\overline{z}^2}{2!} + \;=\; e^{\overline{z}}$$

Then

$$|e^{ix}|^2 \;=\; e^{ix}\overline{e^{ix}} \;=\; e^{ix}e^{-ix} \;=\; e^0 \;=\; 1$$

for real x. ///

[2.2] Trigonometric functions Similarly, $\sin x$ and $\cos x$ both satisfy $f'' = -f$, in *radian measure*: making this differential equation hold determines what radian measure must be. The two trig functions are distinguished from each other by the initial conditions

$$\cos 0 \;=\; 1,\; \cos' 0 \;=\; 0 \qquad \text{and} \qquad \sin 0 \;=\; 0,\; \sin' 0 \;=\; 1$$

Presuming existence of power series expansions $f(x) = c_0 + c_1x + c_2x^2 + \ldots$ for solutions of $f''(x) = -f(x)$ and differentiating term-wise, the differential equation gives

$$2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 \ldots \;=\; -\Big(c_0 + c_1x + c_2x^2 + \ldots\Big)$$

so

$$n(n-1)c_n \;=\; -c_{n-2}$$

The first two coefficients c_0, c_1 determine all: $c_{2n} = (-1)^n c_0/(2n)!$ and $c_{2n+1} = (-1)^n c_1/(2n+1)!$. Thus,

$$\cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \qquad \sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

These power series do converge for all $x \in \mathbb{R}$ and, in fact, for $x \in \mathbb{C}$, so really do produce solutions to the differential equations.

[2.3] Euler's identity The power series for e^x, $\cos x$, and $\sin x$ suggest Euler's identity

$$e^{ix} \;=\; \cos x + i \sin x$$

at least for *real* x, and then for *complex* x, extending the definition of cosine and sine to complex numbers by their power series expansions:

$$e^{ix} \;=\; 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots \;=\; 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \ldots$$

$$= \;\Big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\Big) + i\Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\Big) \;=\; \cos x + i \sin x$$

It is amusing to use Euler's relation, coming from power series, to prove identities seemingly related to triangles and circles:

[2.3.1] Corollary: For real or complex z

$$\cos^2 z + \sin^2 z \;=\; 1$$

Proof: For z real or complex,

$$1 \;=\; e^{iz}e^{-iz} \;=\; (\cos z + i \sin z)(\cos z - i \sin z) \;=\; \cos^2 z + \sin^2 z$$

as desired. ///

Information about π, implicit in the differential equations satisfied by $\cos x$ and $\sin x$ in radian measure, can be certified to give the same π we thought it should:

[2.3.2] Lemma: The function $\cos x$ defined by the previous power series has least positive zero (namely, $\pi/2$) between $\frac{3}{2}$ and 2.

Proof: From elementary estimates: noting that everything is *real*,

$$\cos\left(\frac{3}{2}\right) \;=\; 1 - \frac{\left(\frac{3}{2}\right)^2}{2!} + \frac{\left(\frac{3}{2}\right)^4}{4!} - \frac{\left(\frac{3}{2}\right)^6}{6!} + \ldots \;\geq\; 1 - \frac{1}{2} \;>\; 0$$

Indeed, further, for any $0 \leq x \leq \frac{3}{2}$, a similar inequality proves that $\cos x > 0$, so is non-zero. Meanwhile,

$$\cos 2 \;=\; 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \ldots \;\leq\; 1 - \frac{4}{2} + \frac{16}{24} \;<\; 0$$

The intermediate value theorem promises a zero between $\frac{3}{2}$ and 2. ///

[2.3.3] Corollary:

$$e^{\pi i} = -1 \qquad e^{\pi i/2} = i \qquad \text{and} \qquad e^{z+2\pi i} = e^z$$

for all complex z. ///

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3. *Differentiating power series: Abel's theorem*

[3.0.1] Theorem: (Abel) A function f expressible as a power series $f(z) = \sum_{n\geq 0} c_n\,(z-z_o)^n$ absolutely convergent for $|z-z_o| < r$ is *differentiable* for $|z-z_o| < r$. The derivative is obtained by differentiating term-wise, giving the expected expression

$$f'(z) \;=\; \sum_{n\geq 0} nc_n\,(z-z_o)^{n-1} \qquad\qquad (\text{absolutely convergent for } |z-z_o| < r)$$

[3.0.2] Corollary: Differentiating repeatedly,

$$f^{(k)}(z) \;=\; \sum_{n\geq 0} n(n-1)\ldots(n-k+1)\,c_n\,(z-z_o)^{n-k}$$

In particular, the power series coefficients are *uniquely determined* by the function's derivatives' values, namely, $c_k = f^{(k)}(z_o)/k!$. ///

Proof: Without loss of generality, $z_o = 0$. Fix $0 < \rho < r$, and $|\zeta| < \rho$. For $|z| < r$, the power series

$$g(z) \;=\; \sum_{n\geq 0} nc_n\,z^{n-1}$$

is demonstrably absolutely convergent from the convergence of the original. Consider

$$\frac{f(z)-f(\zeta)}{z-\zeta} - g(\zeta) \;=\; \sum_{n\geq 1} c_n\left(\frac{z^n-\zeta^n}{z-\zeta} - n\zeta^{n-1}\right)$$

At $n = 1$, the parenthetical expression is 1. For $n > 1$, it is

$$(z^{n-1} + z^{n-2}\zeta + z^{n-3}\zeta^2 + \ldots + z\zeta^{n-2} + \zeta^{n-1}) - n\zeta^{n-1}$$
$$= (z^{n-1}-\zeta^{n-1})+(z^{n-2}\zeta-\zeta^{n-1})+(z^{n-3}\zeta^2-\zeta^{n-1})+\ldots+(z^2\zeta^{n-3}-\zeta^{n-1})+(z\zeta^{n-2}-\zeta^{n-1})+(\zeta^{n-1}-\zeta^{n-1})$$
$$= (z-\zeta)\Big[(z^{n-2}+\ldots+\zeta^{n-2}) + \zeta(z^{n-3}+\ldots+\zeta^{n-3}) + \ldots + \zeta^{n-3}(z+\zeta) + \zeta^{n-2} + 0\Big]$$
$$= (z-\zeta)\sum_{k=0}^{n-2}(k+1)\, z^{n-2-k}\,\zeta^k$$

For $|z|$ and $|\zeta|$ both smaller than ρ, the latter sum is dominated by

$$|z-\zeta|\,\rho^{n-2}\,\frac{n(n-1)}{2} \;<\; n^2\,|z-\zeta|\,\rho^{n-2}$$

Thus,

$$\left|\frac{f(z)-f(\zeta)}{z-\zeta} - g(\zeta)\right| \;\le\; |z-\zeta|\cdot\sum_{n\ge 2}|c_n|\,n^2\,\rho^{n-2}$$

Since $\rho < r$ the latter series converges absolutely, so the left-hand side goes to 0 as $z \to \zeta$. ///

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4. *Path integrals*

The idea of an integral $\int_\gamma f$ of a function along a path γ in an open subset Ω of $\mathbb{C}$ is an extension of the idea of an integral on an interval $[a,b] \subset \mathbb{R}$, and expressible as a limit of Riemann sums. The limit must be shown to exist. In fact, path integrals are *expressible* as integrals on intervals, as below.

Simultaneously, we expect a relation to complex differentiation, extending the fundamental theorem of single-variable calculus: when $f = F'$ for complex-differentiable F on open set Ω, it should be that, for any path γ from z_1 to z_2 inside Ω,

$$\int_\gamma F' \;=\; F(z_2) - F(z_1)$$

Proof of this will reduce to the single-variable calculus situation.

[4.1] Riemann-sum version of *path integral* Given a smooth curve γ connecting two points z_1 and z_2 in an open set $\Omega \subset \mathbb{C}$, and a continuous $\mathbb{C}$-valued function f on Ω, one natural notion of $\int_\gamma f$ would be as a limit of Riemann sums, as follows. Given $\delta > 0$, choose points $w_1 = z_1, w_2, w_3, \ldots, w_{n-1}, w_n = z_2$ on γ so that $|w_j - w_{j+1}| < \delta$, and form the Riemann sum

$$f(w_1)\cdot(w_2-w_1)+f(w_2)\cdot(w_3-w_2)+f(w_3)\cdot(w_3-w_2)+\ldots+f(w_{n-2})\cdot(w_{n-1}-w_{n-2})+f(w_{n-1})\cdot(w_n-w_{n-1})$$

The limit of such sums as $\delta \to 0$ for continuous f and for sufficiently nice paths γ exists, for reasons similar to that for ordinary Riemann integrals. In fact, we prove this by reducing to the single-variable calculus situation, as follows.

[4.1.1] Remark: The choice of $w_j - w_{j-1}$ rather than $w_{j-1} - w_j$ is meant to suggest that the *direction* of the path integral is from z_1 to z_2. Indeed, writing $-\gamma$ for the path γ traversed in the opposite direction, almost by definition

$$\int_{-\gamma} f \;=\; -\int_\gamma f$$

[4.2] Path integrals on parametrized paths A *parametrized path* γ in an open set $\Omega \subset \mathbb{C}$ is a *nice* function $\gamma : [a,b] \to \Omega$ for some interval $[a,b] \subset \mathbb{R}$. Here *nice* probably means *piecewise-differentiable*: γ is *continuous* throughout $[a,b]$, and $[a,b]$ breaks into finitely-many subintervals on each of which γ is continuously differentiable.

[4.2.1] Proposition: For continuous, complex-valued f on Ω and nice parametrized path $\gamma : [a,b] \to \Omega$, the path integral $\int_\gamma f$ of f along γ expressed as a limit of Riemann sums, is expressible in terms of the parametrization, as

$$\int_\gamma f \;=\; \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \qquad\qquad (\text{where } \gamma'(t) = \tfrac{d}{dt}\gamma(t) \text{ as expected})$$

Proof: The point is that Riemann sums directly on the curve are equal to Riemann sums on $[a,b]$, via the parametrization. The factor $\gamma'(t)$ is the limiting case of the multiplication by differences $w_j - w_{j-1}$ in the direct Riemann sum version.

Consider $\gamma(t) = x(t) + iy(t)$ with once-continuously-differentiable real-valued functions $x(t), y(t)$. Given $\delta > 0$, choose $a = t_1 < t_2 < \ldots < t_n = b$ on $[a,b]$ such that $|\gamma(t_j) - \gamma(t_{j-1})| < \delta$, using the *uniform* continuity of γ on the bounded interval $[a,b]$. The mean value theorem applied to functions $x(t), y(t)$ shows that $\gamma(t_{j+1}) - \gamma(t_j)$ is well approximated by $\gamma'(t_j)(t_{j+1} - t_j)$:

$$\frac{\gamma(t_j) - \gamma(t_{j-1})}{t_{j+1} - t_j} - \gamma'(t_j) \;\to\; 0 \qquad\qquad (\text{as } t_{j+1} - t_j \to 0)$$

Again because $[a,b]$ is *bounded*, this limit behavior is *uniform*: given $\varepsilon > 0$, there is $\eta > 0$ such that

$$\Big|\frac{\gamma(t) - \gamma(\tau)}{t - \tau} - \gamma'(\tau)\Big| \;<\; \eta \qquad\qquad (\text{for } t \neq \tau \text{ in } [a,b] \text{ with } |t - \tau| < \eta)$$

Thus, the direct Riemann sum is well approximated by a modified form:

$$\Big|\sum_j f(\gamma(t_j))(\gamma(t_{j+1}) - \gamma(t_j)) \;-\; \sum_j f(\gamma(t_j))\gamma'(t_j)(t_{j+1} - t_j)\Big| \;<\; \eta \cdot \sum_j \big|f(\gamma(t_j))\big| \cdot (t_{j+1} - t_j)$$

The modified Riemann sum $\sum_j f(\gamma(t_j))\gamma'(t_j)(t_{j+1} - t_j)$ is exactly a Riemann sum for the parametrized-path integral. The right-hand side in the inequality is η times a Riemann sum for the real-valued function $t \to |f(\gamma(t))|$, and these Riemann sums converge to a finite number. Since η is as small as desired, the left hand side goes to 0. Thus, Riemann sums for the parametrized-path integral converge to the same limit as the Riemann sums for the directly-defined path integral. ///

[4.2.2] Remark: The previous discussion also shows that the path integral does not depend on the parametrization. Independence of path parametrization can also be proven directly by changing variables, from the *chain rule*:

Let $\gamma_2 : [a_2, b_2] \to \Omega$ and $\varphi : [a_2, b_2] \to [a,b]$ differentiable such that $\gamma \circ \varphi = \gamma_2$. Unwinding the definitions, and using the chain rule, with $u = \varphi(t)$,

$$\int_{\gamma_2} f \;=\; \int_{a_2}^{b_2} f(\gamma_2(t))\,\gamma_2'(t)\,dt \;=\; \int_{a_2}^{b_2} f(\gamma \circ \varphi(t))\,(\gamma \circ \varphi)'(t)\,dt$$

$$=\; \int_{a_2}^{b_2} f(\gamma \circ \varphi(t))\,\gamma'(\varphi(t))\,d\varphi(t) \;=\; \int_a^b f(\gamma(u))\,\gamma'(u)\,du \;=\; \int_\gamma f$$

proving independence of parametrization.

[4.3] Example From Euler's identity, the unit circle can be parametrized by $\gamma(t) = e^{it}$ with $t \in [0, 2\pi]$. For integers n,

$$\int_\gamma z^n \;=\; \int_0^{2\pi} (e^{it})^n \,\frac{de^{it}}{dt}\, dt \;=\; \int_0^{2\pi} e^{nit}\, ie^{it}\, dt \;=\; \int_0^{2\pi} i\, e^{(n+1)it}\, dt \;=\; \begin{cases} \Big[i\Big]_0^{2\pi} = 2\pi i & (\text{for } n = -1) \\ \Big[\dfrac{i\, e^{(n+1)it}}{(n+1)i}\Big]_0^{2\pi} = 0 & (\text{for } n \neq -1) \end{cases}$$

[4.4] Primitives and independence of path

Path integrals and complex differentiation have the same relation as the fundamental theorem of calculus gives for integrals and derivatives on intervals, namely, $\int_a^b F' = F(b) - F(a)$.

For f on Ω, a complex-differentiable function F on Ω with $F' = f$ is a *primitive* of f. For any parametrization $\gamma : [a, b] \to \Omega$ of a path from z_1 to z_2, by the usual fundamental theorem of calculus,

$$\int_\gamma F' \;=\; \int_a^b F'(\gamma(t))\, \gamma'(t)\, dt \;=\; \int_a^b \frac{d}{dt} F(\gamma(t))\, dt \;=\; F(\gamma(b)) - F(\gamma(a)) \;=\; F(z_2) - F(z_1)$$

Thus, when f has a primitive, any path integral of f only depends on the endpoints.

A path $\gamma : [a, b] \to \Omega$ is *closed* when $\gamma(a) = \gamma(b)$. When f has a primitive, its integral over any closed path is 0.

[4.5] Continuity

When two paths γ_1, γ_2 are *sufficiently close*, for any continuous f the integrals $\int_{\gamma_1} f$ and $\int_{\gamma_2} f$ should be close.

Perhaps surprisingly, the two curves must be close *not only* in the sense that the sets $\gamma_1[a, b]$ and $\gamma_2[a, b]$ are suitably close, but also the derivatives γ_1' and γ_2' must be close. More precisely, for both paths map from $[a, b]$, and if $|\gamma_1(t) - \gamma_2(t)| < \varepsilon$ *and* $|\gamma_1'(t) - \gamma_2'(t)| < \varepsilon'$ for all $t \in [a, b]$, then, using the identity $aA - bB = a(A - B) + (a - b)B$,

$$\Big| \int_{\gamma_1} f - \int_{\gamma_2} f \Big| \;\le\; \int_a^b \big| f(\gamma_1(t))\gamma_1'(t) - f(\gamma_2(t))\gamma_2'(t) \big|\, dt$$

$$\le\; \int_a^b \big|f(\gamma_1(t))\big| \cdot \big|\gamma_1'(t) - \gamma_2'(t)\big|\, dt + \int_a^b \big|f(\gamma_1(t)) - f(\gamma_2(t))\big| \cdot \big|\gamma_2'(t)\big|\, dt \;<\; \varepsilon' \cdot |b - a| \cdot C + \varepsilon \cdot |b - a| \cdot C'$$

where C is the maximum of the continuous function $|f|$ on a compact region containing γ_1, γ_2, and C' is the maximum of $|\gamma_2'|$ on such a region. Thus, making $\varepsilon, \varepsilon'$ small makes the difference of path integrals small.

[4.6] Approximations by polygons

Cauchy's theorem, below, proves a fundamental property of path integrals over triangles. The corresponding fundamental result follows for polygons. The result for smooth curves, and finitely-piecewise continuously differentiable curves, requires expression of these curves as suitable limits of polygons.

The original limit-of-Riemann-sum definition of a path integral, and expression as parametrized-path integral, almost accomplished this.

Again, given a smooth curve γ connecting two points z_1 and z_2 in an open set $\Omega \subset \mathbb{C}$, and a continuous $\mathbb{C}$-valued function f on Ω, $\int_\gamma f$ is a limit of Riemann sums. For $\delta > 0$, choose points $w_1 = z_1, w_2, w_3, \ldots, w_{n-1}, w_n = z_2$ on γ so that $|w_j - w_{j+1}| < \delta$, and form the Riemann sum as earlier:

$$f(w_1) \cdot (w_2 - w_1) + f(w_2) \cdot (w_3 - w_2) + \ldots + f(w_{n-2}) \cdot (w_{n-1} - w_{n-2}) + f(w_{n-1}) \cdot (w_n - w_{n-1})$$

By *uniform* continuity of f on an open set with compact closure containing the path, given $\varepsilon > 0$, for δ small enough, $|f(z) - f(w_{j-1})| < \varepsilon$ for all z on the straight line segment ℓ_j from w_{j-1} to w_j, so

$$\Big| \int_{\ell_j} f \;-\; f(w_{j-1}) \cdot \int_{\ell_j} 1 \Big| \;<\; \varepsilon \cdot |w_j - w_{j-1}|$$

and

$$\Big| \int_\gamma f \;-\; \sum_j \int_{\ell_j} f \Big| \;<\; \varepsilon \cdot \sum_j |w_j - w_{j-1}|$$

Obviously, the straight line segments ℓ_j assemble to a polygon approximating γ. The situation suggests that the limit as $\delta \to 0^+$ of $\sum_j |w_j - w_{j-1}|$ is the *length* of γ. This will follow from the finitely-piecewise continuous differentiability of γ.

It suffices to consider one of the finitely-many continuously differentiable pieces of γ, thus, we take $\gamma : [a, b] \to \Omega$ continuously differentiable without loss of generality. We claim that

$$\lim_{\delta \to 0} \sum_j |w_j - w_{j-1}| \;=\; \int_a^b |\gamma'(t)|\, dt$$

With $\gamma(t_j) = w_j$, by the uniform continuity of the derivative,

$$\frac{w_j - w_{j-1}}{t_j - t_{j-1}} - \gamma'(t_{j-1}) \;\longrightarrow\; 0 \qquad\qquad \text{(uniformly, as } \delta \to 0)$$

Thus, for given $\varepsilon > 0$, for small enough $\delta > 0$,

$$\Big| \sum_j |w_j - w_{j-1}| - \sum_j |\gamma'(t_{j-1})| \cdot (t_j - t_{j-1}) \Big| \;<\; \varepsilon \cdot \sum_j |t_j - t_{j-1}| \;=\; \varepsilon \cdot |b - a| \;\longrightarrow\; 0$$

The Riemann sum involving γ' goes to $\int_a^b |\gamma'(t)|\, dt$. ///

[4.6.1] Remark: More generally, curves γ for which the limit of the sum $\sum_j |w_j - w_{j-1}|$ exists are *rectifiable*. There do exist *continuous* but *not-rectifiable* curves. We need at-worst finitely-piecewise continuously differentiable curves, so worry about further possibilities is not necessary.

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5. *Cauchy's theorem*

The theorem is that the path integral of a complex-differentiable complex-valued function f on a region Ω, over a path γ that can be contracted to a point *inside* Ω, is 0. Goursat eliminated superfluous further hypotheses.

The base case is γ the boundary of a *triangle* sitting inside Ω, traced counter-clockwise.

[5.0.1] Theorem: $\int_\gamma f = 0$.

Proof: Subdivide the given triangle T_0 by connecting the midpoints of the sides of the given triangle T_0 to each other, forming four similar half-sized triangles. The path integral of f over the whole triangle is the sum of the path integrals over the four similar triangles, since the added-on paths are traced in both directions, so cancel. Of these four, let T_1 be the triangle maximizing the absolute value of the path integral around it.

Subdivide T_1 similarly into four similar half-sized triangles, and of these let T_2 be the triangle maximizing the absolute value of the path integral around it. Continue.

The nested triangles $T_0 \supset T_1 \supset T_2 \supset \ldots$ have a unique intersection point z_o, since the diameter of T^n is 2^{-n} times the diameter of T_0. Indeed, any choice of points $w_n \in T_n$ is a Cauchy sequence, since the diameters are repeatedly halved, and every such choice has the same limit, for the same reason.

Let γ_n be the path integral around T_n, counter-clockwise. Complex-differentiability of f at z_o asserts that, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\Big|f(z) - f(z_o) - f'(z_o)(z - z_o)\Big| \;<\; \varepsilon \cdot |z - z_o| \qquad\qquad \text{(for all } z \text{ with } |z - z_o| < \delta)$$

Let $r_n = \max_{z\in\gamma_n} |z - z_o|$ denote the *radius* of γ_n about z_o. Given $\varepsilon > 0$, choose n sufficiently large so that $r_n < \delta$. Then

$$\Big|\int_{\gamma_n} f(z) - f(z_o) - f'(z_o)(z - z_o)\; dz\Big| \;<\; \varepsilon \cdot r_n \cdot \text{length } \gamma_n \;=\; \varepsilon \cdot (2^{-n}\, r_0) \cdot (2^{-n} \cdot \text{length } \gamma) \qquad \text{(for } z \in \gamma_n)$$

It is easy to check that the path integrals $\int_\gamma f$ of constants and linear functions f around triangles are 0, since it is easy to find *primitives* for such f: $1 = (z)'$ and $z = (z^2/2)'$. For a triangular path γ' with vertices a, b, c and F a primitive for a given function,

$$\int_{\gamma'} F'\; dz \;=\; \Big(F(b) - F(a)\Big) + \Big(F(c) - F(b)\Big) + \Big(F(a) - F(c)\Big) \;=\; 0$$

Thus,

$$\Big|\int_{\gamma_n} f\;\Big| \;<\; 4^{-n} \cdot \varepsilon \cdot r_0 \cdot \text{length } \gamma \qquad\qquad \text{(for } z \in \gamma_n)$$

At each step, T_n maximized the absolute value of the path integral, so

$$\Big|\int_\gamma f\Big| \;\;\le\;\; 4^n \cdot \Big|\int_{\gamma_n} f\Big| \;\;<\;\; 4^n \cdot 4^{-n} \cdot \varepsilon \cdot r_0 \cdot \text{length } \gamma \;\;=\;\; \varepsilon \cdot \big(r_0 \cdot \text{length } \gamma\big)$$

This holds for every $\varepsilon > 0$, so $\int_\gamma f = 0$. ///

[5.0.2] Corollary: For a polygon P in a *convex* set Ω and γ the path integral around P traced counter-clockwise, for f complex-differentiable on Ω, $\int_\gamma f = 0$.

Proof: Pick some point z_o in Ω, and form triangles from z_o and every pair of consecutive vertices of P. These triangles lie inside Ω since it is convex. The sum of the (counter-clockwise) integrals over these triangles is the integral over Γ, since the added-on paths are traced in both directions, so cancel. The integral over each triangle is 0, by Cauchy's theorem. ///

[5.0.3] Corollary: For a finitely-piecewise continuously differentiable *closed* curve γ in a *convex* set Ω and γ the path integral around P traced counter-clockwise, for f complex-differentiable on Ω, $\int_\gamma f = 0$.

Proof: Approximate γ by polygons inside Ω. ///

[5.0.4] Remark: Unsurprisingly, the same argument works under a weaker hypothesis than convexity: for Ω and γ *starlike* about z, meaning that the line segment connecting z to any other point of Ω lies entirely inside Ω, and the line segment connecting z to any point of γ meets γ only at that point.

[5.0.5] Remark: Ever-more complicated, weaker hypotheses on the topology of γ and Ω still allow the conclusion $\int_\gamma f = 0$. A simple useful case is that γ is *contractible* in Ω, meaning that it can be (piece-wise smoothly!) shrunk down to a point without passing outside Ω.

6. *Cauchy's formula/integral representation*

Again, the base case involves the very simplest paths, for example, triangles:

[6.0.1] Theorem: For f complex differentiable near z, for γ a counter-clockwise path around a triangle having z in its interior,

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

Proof: The function

$$F(z) \;=\; \frac{f(z) - f(z_o)}{z - z_o}$$

is complex-differentiable where f is, except possibly z_o. Let γ' be the path counterclockwise around a small triangle T' about z_o, entirely inside the larger triangle T. Connect the vertices of T' to those of T. As in earlier episodes, the sum of path integrals over the boundaries of the three resulting quadrilaterals and the boundary γ' of T' is the integral over γ, because the interior paths are traversed in both directions, so cancel.

Cauchy's theorem for nice polygons above shows that the integral over each quadrilateral is 0. Thus,

$$\int_\gamma F \;=\; \int_{\gamma'} F$$

Using *continuity* of F at z_o, given $\varepsilon > 0$ there is $\delta > 0$ such that $|F(z) - F(z_o)| < \varepsilon$ for $|z - z_o| < \delta$. With T' chosen small enough to be inside the disk of radius δ at z_o,

$$\Big|\int_{\gamma'} F(z) - F(z_o)\,dz\Big| \;<\; \varepsilon\cdot\text{length }\gamma' \;\le\; \varepsilon\cdot 6\delta$$

Again, the integral of the constant $F(z_o)$ around a closed path is 0. Thus, the integral of F itself is smaller than every $\varepsilon > 0$, and is necessarily 0. Thus, relabelling the variables to better express our intent,

$$0 \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta) - f(z)}{\zeta - z}\,d\zeta \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \;-\; f(z)\cdot\frac{1}{2\pi i}\int_\gamma \frac{1}{\zeta - z}\,d\zeta$$

Again, the value of the integral of $1/(\zeta - z)$ around T is equal to that counter-clockwise around a small triangle T' enclosing z.

There are many ways to show that this integral is $2\pi i$. One way is to put an even-smaller circular path σ around z, and connect it to the small triangle by lines, cutting the part of T' outside σ into convex regions. Applying Cauchy's theorem to the complex-differentiable function $\zeta \to \frac{1}{\zeta - z}$ on each of these, as in other episodes, the integral around T' is equal to that around σ. Parametrize $\sigma : [0, 2\pi] \to \Omega$ by $\sigma(t) = z + r\cdot e^{it}$, so, with fixed z,

$$\int_\sigma \frac{d\zeta}{\zeta - z} \;=\; \int_0^{2\pi} \frac{d(z + re^{it}}{(z + re^{it}) - z} \;=\; \int_0^{2\pi} \frac{ire^{it}\,dt}{re^{it}} \;=\; \int_0^{2\pi} i\,dt \;=\; 2\pi i$$

This proves Cauchy's formula for triangles. ///

[6.0.2] Corollary: Complex-differentiable f is *infinitely* differentiable, and

$$f'(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta$$

$$f''(z) \;=\; \frac{2}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^3}\,d\zeta$$

$$f'''(z) \;=\; \frac{3!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^4}\,d\zeta$$

and, generally,

$$f^{(n)}(z) \;=\; \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^{(n+1)}}\,d\zeta$$

Proof: The integral in the Cauchy formula *on a small triangle* around a given point is infinitely-differentiable with respect to z, so f is. Differentiating under the integral with respect to z gives the formulas for the derivatives.

Differentiating under the integral is obviously necessary, and immediately gives the conclusion. In this particular example, because the dependence of the integrands on z is so simple, it is not hard to justify. Namely, we guess the obvious expression for the derivative, and check the definition: first,

$$\frac{\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-(z+h)}\,d\zeta - \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta-z}\,d\zeta}{h} \;-\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma f(\zeta)\cdot\left(\frac{\frac{1}{\zeta-(z+h)} - \frac{1}{\zeta-z}}{h} - \frac{1}{(\zeta-z)^2}\right)d\zeta$$

The rational expression has the expected rearrangement

$$\frac{\frac{1}{\zeta-(z+h)} - \frac{1}{\zeta-z}}{h} - \frac{1}{(\zeta-z)^2} \;=\; \frac{(\zeta-z)-(\zeta-(z+h))}{(\zeta-(z+h))(\zeta-z)\cdot h} - \frac{1}{(\zeta-z)^2} \;=\; \frac{1}{(\zeta-(z+h))(\zeta-z)} - \frac{1}{(\zeta-z)^2}$$

$$= \frac{(\zeta-z)-(\zeta-(z+h))}{(\zeta-(z+h))(\zeta-z)^2} \;=\; \frac{h}{(\zeta-(z+h))(\zeta-z)^2}$$

As $h \to 0$, for z uniformly bounded away from ζ, this goes to 0 uniformly in z, ζ. Since f is continuous, it is uniformly bounded on the compact set consisting of the curve γ. Thus, the definition of the derivative being given by the expected formula is verified. ///

Weakening a convexity hypothesis: for the following corollary, a region Ω is *starlike* about z when the line segment connecting z to any other point in Ω lies entirely inside Ω. A path γ inside starlike Ω is *starlike* about z when the line segment connecting z to any point w on γ meets γ only at w.

[6.0.3] Corollary: For γ a starlike polygonal path about z, traced counter-clockwise, and f complex-differentiable on a starlike Ω containing γ,

$$\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta-z} \;=\; 2\pi i\cdot f(z)$$

Proof: Put a small triangle T around z, small enough so that, by continuity, reasonable choices of line segments connecting the vertices to the vertices of the polygon lie inside Ω. The sum of the integrals over the resulting triangles other than T are 0, by Cauchy's theorem, and the integral around T gives $2\pi i\, f(z)$, as just proven. ///

[6.0.4] Corollary: For γ a starlike finitely-piecewise continuously differentiable path about z, traced counter-clockwise, and f complex-differentiable on a starlike Ω containing γ,

$$\int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta - z} \;=\; 2\pi i \cdot f(z)$$

Proof: Approximate γ by convex polygonal paths. ///

[6.0.5] Remark: The same arguments show in quite general situations that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta - z} \;=\; f(z)\cdot \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z}$$

The latter integral is proven, in various ways, to be an integer, and is the *winding number* of γ around z, which is meant to be the number of times γ *goes around* z. This is imprecise as it stands, but can be made precise in various ways, the best ones involving a little algebraic topology.

A *simple closed curve* γ about z is one such that

$$\frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z} \;=\; 1$$

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7. Power series expansions, Morera's theorem

[7.0.1] Theorem: A function admitting a Cauchy integral representation

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

for some fixed simple closed path γ about z traced counter-clockwise, *has a convergent power series expansion* for z near every z_o inside γ:

$$f(z) \;=\; \sum_{n=0}^{\infty} \frac{f^{(n)}(z_o)}{n!}\cdot (z - z_o)^n$$

absolutely convergent for $|z - z_o|$ less than the distance from z_o to γ.

[7.0.2] Remark: A function expressible as a convergent power series is called *complex analytic*.

Proof: This is just an expansion of a geometric series, interchange of sum and integral, and invocation of Cauchy's formulas for derivatives. For notational simplicity, take $z_o = 0$ without loss of generality. Then

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta(1-\frac{z}{\zeta})}\, d\zeta \;=\; \frac{1}{2\pi i}\int_\gamma f(\zeta)\frac{1}{\zeta}\Big(1 + \frac{z}{\zeta} + \big(\frac{z}{\zeta}\big)^2 + \ldots\Big)\, d\zeta$$

$$= \sum_{n\geq 0} \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\, d\zeta}{\zeta^{n+1}}\cdot z^n \;=\; \sum_{n\geq 0}\frac{f^{(n)}(0)}{n!}\cdot z^n$$

Interchange of sum and integral is obviously necessary, and can be justified in various fashions. Let r be the minimum distance from z_o to γ. By Cauchy's formulas, with γ parametrized by $[0,1]$,

$$\Big|\frac{f^{(n)}(z_o)}{n!}\Big| \;\le\; \int_0^1 \Big|\frac{f(\gamma(t))\cdot\gamma'(t)}{(\gamma(t)-z_o)^{n+1}}\Big|\,dt \;\le\; \max_\gamma |f|\cdot\max|\gamma'|\cdot\frac{1}{r^{n+1}}$$

Thus, the power series for f in $z-z_o$ is absolutely convergent for $|z-z_o|<r$. ///

[7.0.3] Corollary: A complex differentiable function is infinitely differentiable.

Proof: Abel's theorem. ///

[7.0.4] Note: From here on, we use *complex differentiable* and *complex analytic* as synonyms, and, in fact, replace these by *holomorphic*, in part signifying that we have proved Cauchy's basic results.

[7.0.5] Corollary: *(Morera's theorem)* A continuous function f on an open set Ω with the property that the path integrals $\int_\gamma f$ of f over the boundaries γ of all sufficiently small triangles inside Ω (with interiors inside Ω as well) is *holomorphic*.

Proof: The hypothesis is somewhat stronger than the conclusion of the basic form of Cauchy's theorem about vanishing of path integrals of complex-differentiable functions over triangles. Thus, the proof of Cauchy's integral formula applies to such f, proving

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta-z}$$

for suitable γ enclosing z. Then the above arguments for infinite-differentiability of f and power series expansions follow in the same way. ///

[7.0.6] Corollary: Let $\{f_j\}$ be a sequence of holomorphic functions on an open set Ω, *uniformly convergent on compacts* in the sense that, for every compact $K\subset\Omega$, given $\varepsilon>0$, there is j_o such that for $i,j\ge j_o$

$$\sup_{z\in K}|f_i(z)-f_j(z)| \;<\; \varepsilon$$

Then the pointwise limit $f(z)=\lim_j f_j(z)$ exists and is itself holomorphic.

Proof: The uniform convergence on compacts implies pointwise convergence for each $z_o\in\Omega$, by taking $K=\{z_o\}$ and noting that the sequence $f_j(z_o)$ is Cauchy. Further, the limit $f(z)=\lim_j f_j(z)$ is *continuous*: given $z_o\in\Omega$ and given $\varepsilon>0$, fix a small $K=\{z:|z-z_o|\le r\}$, and choose j large enough so that $|f_j(z_o)-f(z_o)|<\varepsilon$ for all $|z-z_o|\le r$, and $\delta>0$ small enough so that $|f_j(z)-f_j(z_o)|<\varepsilon$ for $|z-z_o|<\delta$. Then

$$|f(z)-f(z_o)| \;\le\; |f(z)-f_j(z)|+|f_j(z)-f_j(z_o)|+|f_j(z_o)-f(z_o)| \;<\; \varepsilon+\varepsilon+\varepsilon$$

If desired, this is easily rearranged to give ε rather than 3ε, proving continuity of the limit f throughout Ω.

With continuity in hand, we can certainly integrate f over boundaries of triangles and other simple closed curves γ inside Ω. Since $\int_\gamma f_j=0$ for all j, and since (the image of) γ is *compact*, the integrals $\int_g amf_j$ go to $\int_\gamma f$, which is therefore 0. By Morera's theorem, f is holomorphic. ///

8. *Identity principle*

[8.0.1] Theorem: If holomorphic functions f, g on a connected open set Ω take the same values at distinct points $z_1, z_2, z_3, \ldots$ in Ω, and $\lim_j z_j = z_o \in \Omega$, then $f = g$ throughout Ω.

Proof: First, one natural line of argument can be followed to its logical end: by continuity of f, g,

$$f(z_o) \;=\; \lim_j f(z_j) \;=\; \lim_j g(z_j) \;=\; g(z_o)$$

Equality of the first derivative at z_o follows similarly:

$$f'(z_o) \;=\; \lim_j \frac{f(z_o) - f(z_j)}{z_o - z_j} \;=\; \frac{g(z_o) - g(z_j)}{z_o - z_j} \;=\; g'(z_o)$$

Perhaps it is feasible to express higher derivatives at z_o as more complicated iterated difference quotients, but this is best done in a somewhat repackaged form: consider $h = f - g$, a holomorphic function with $h(z_j) = 0$ and $z_j \to z_o$. That is, inside every punctured disk $0 < |z - z_o| < \delta$ there is a zero of h. If h were not *identically* 0, it would have a convergent power series

$$h(z) \;=\; c_N(z - z_o)^N + c_{N+1}(z - z_o)^{N+1} + \ldots \qquad\qquad (\text{with } c_N \neq 0)$$

The idea is that for z very close to z_o the $(z - z_o)^N$ dominates the power series, but is not 0 for $z \neq z_o$, contradicting the assumption that h is not identically 0. Indeed, for some $r > 0$ the power series is absolutely convergent for $|z - z_o| \leq r$, so certainly $c_n r^n \to 0$, so these numbers are bounded in absolute value, say by C. For $|z_j - z_o| = \delta$, with $0 < \delta < r$ and $\delta < \frac{1}{2}$,

$$\Big| \sum_{n \geq N+1} c_n (z_j - z_o)^n \Big| \;\leq\; C \sum_{n \geq N+1} \delta^n \;\leq\; \frac{C \cdot \delta^{N+1}}{1 - \delta} \;\leq\; 2C \cdot \delta^{N+1}$$

Meanwhile,

$$|c_N (z_j - z_o)^N| \;=\; |c_N| \cdot \delta^N$$

Taking j large enough such that δ is small enough so that $|c_N| \cdot \delta^N > 2C \cdot \delta^{N+1}$, we have

$$|h(z_j)| \;=\; \Big| \sum_n c_n (z_j - z_o)^n \Big| \;\geq\; |c_N| \cdot \delta^N - 2C \cdot \delta^{N+1} \;>\; 0$$

contradicting $h(z_j) = 0$. Thus, $h = f - g$ must have been identically 0. ///

[8.0.2] Example: Euler's integral for the *Gamma function* is

$$\Gamma(s) \;=\; \int_0^\infty t^s \, e^{-t} \, \frac{dt}{t} \qquad\qquad (\text{for } \mathrm{Re}(s) > 0)$$

For $x > 0$, by changing variables,

$$\int_0^\infty t^s \, e^{-tx} \, \frac{dt}{t} \;=\; x^{-s} \int_0^\infty t^s \, e^{-t} \, \frac{dt}{t} \;=\; x^{-s} \cdot \Gamma(s)$$

For $y \in \mathbb{R}$, with $z = x + iy$, consider

$$f(z) \;=\; f(x + iy) \;=\; \int_0^\infty t^s \, e^{-t(x+iy)} \, \frac{dt}{t} \;=\; \int_0^\infty t^s \, e^{-tz} \, \frac{dt}{t}$$

It no longer makes sense to change variables. The holomorphy of $f(z)$ for $\mathrm{Re}(z) > 0$ follows from Morera's theorem, for example. On the other hand, let

$$g(z) \;=\; (x+iy)^{-s} \cdot \Gamma(s) \;=\; z^{-s} \cdot \Gamma(s)$$

with some sense of z^{-s} deserving further discussion later. We have shown that $f(x) = g(x)$ for all $x > 0$. Presuming that $g(z)$ is holomorphic, the identity principle gives an outcome as though change-of-variables were legitimate:

$$z^{-s} \cdot \Gamma(s) \;=\; g(z) \;=\; f(z) \;=\; \int_0^\infty t^s\, e^{-tz}\, \frac{dt}{t} \qquad\qquad (\text{for } \mathrm{Re}(z) > 0)$$

Indeed, writing

$$z^{-s} \;=\; \Gamma(s)^{-1} \int_0^\infty t^s\, e^{-tz}\, \frac{dt}{t} \qquad\qquad (\text{for } \mathrm{Re}(z) > 0)$$

apparently defines an s^{th} power of z.

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9. *Liouville's theorem: bounded entire functions are constant*

Among other applications, by this point we can prove that the complex numbers are *algebraically closed*, that is, that every non-constant polynomial with complex coefficients has a complex zero, as a corollary of Liouville's theorem, itself a corollary of Cauchy's results.

A holomorphic function defined on the entire complex plane is called *entire*.

[9.0.1] Theorem: *(Liouville)* A bounded entire function is constant.

Proof: The power series expansion of entire and bounded f converges for all $z \in \mathbb{C}$, since the function is entire.

With $|f| \le C$, from Cauchy's formula integrating counter-clockwise over a large circle of radius $R \ge 2|z|$,

$$|f^{(n)}(z)| \;=\; \Big|\frac{n!}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)\,d\zeta}{(\zeta - z)^{n+1}}\;\Big| \;\le\; \frac{C}{2\pi}\int_0^{2\pi} \Big|\frac{iRe^{it}}{(Re^{it}-z)^{n+1}}\Big|\,dt \;\le\; \frac{C}{2\pi}\cdot 2\pi \frac{R}{(R/2)^{n+1}} \;\le\; 2^{n+1} C \cdot R^{-n}$$

This is true for all R meeting the condition $R \ge 2|z|$. As $R \to \infty$, for $n > 0$, this goes to zero. That is, the power series expansion of f consists of the 0^{th} term only, so f is constant. ///

[9.0.2] Corollary: A non-constant polynomial $P(z)$ has a complex zero.

Proof: Without loss of generality, suppose P is *monic*: $P(z) = z^n + c_{n-1}z^{n-1} + \ldots + c_0$ had no 0, and let $f(z) = 1/P(z)$. By the usual quotient rule, and so on, f is complex-differentiable. Since

$$|P(z)| \;\ge\; |z|^n - |z|^{n-1} \cdot \big(|c_{n-1}| + \ldots + |c_0|\big)$$

$|P(z)| \ge |z|^n/2$ for $|z|$ larger than $R = 2(|c_{n-1}| + \ldots + |c_0|)$, and $|f(z)| \le 1/|z|$ for $|z| \ge R$. On the compact set $|z| \le R$ the continuous function f is bounded. Thus, the complex-differentiable function f is entire and bounded, so is constant, by Liouville. This is impossible, since P is of positive degree. ///

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10. *Laurent expansions around singularities*

The basic Cauchy theory is disk-oriented, since power series converge in disks. The next simplest region from this viewpoint is an *annulus* of inner radius r, outer radius R, about a point z_o:

$$\Omega \;=\; \{z \in \mathbb{C} \;:\; r < |z - z_o| < R\}$$

[10.1] Laurent expansions on an annulus

[10.1.1] Theorem: A holomorphic function f in the annulus Ω has a *Laurent expansion*

$$f(z) \;=\; \sum_{n\in\mathbb{Z}} c_n\,(z-z_o)^n \;=\; \ldots + c_{-2}(z-z_o)^{-2} + c_{-1}(z-z_o)^{-1} + c_0 + c_1(z-z_o) + c_2(z-z_o)^2 + \ldots$$

absolutely convergent in the annulus, uniformly on compact subsets. The Laurent coefficients c_n are given by

$$c_n \;=\; \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_R} \dfrac{f(\zeta)\,d\zeta}{(\zeta-z_o)^{n+1}} & (\text{for } n \ge 0) \\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_r} \dfrac{f(\zeta)\,d\zeta}{(\zeta-z_o)^{-n+1}} & (\text{for } n < 0) \end{cases}$$

where γ_R any a circle about z_o of radius slightly less than R, and γ_r is a circle about z_o of radius slightly more than r. These coefficients are *unique*, for $r < |z-z_o| < R$.

[10.1.2] Remark: The positive-index terms give a power series convergent at least in $|z-z_o| < R$, and the negative-index terms give a power series in $(z-z_o)^{-1}$ convergent at least in $|z-z_o| > r$.

Proof: Let γ be the path that first traverses γ_R, then to γ_r along a radial segment toward z_o, traverses γ_r backward, then back out to γ_R along the same radial segment. The two integrals along the radial segment are in opposite directions, so cancel each other, giving

$$\int_{\gamma_R} f - \int_{\gamma_r} f \;=\; \int_\gamma f$$

Further, for z between γ_R and γ_r, Cauchy's integral formula gives

$$f(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} \;=\; \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(\zeta)\,d\zeta}{\zeta - z} \;-\; \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)\,d\zeta}{\zeta - z}$$

The integral over γ_R can be rearranged just as was done in the discussion of Cauchy's formula for derivatives and power series expansions on a disk, producing the non-negative-index terms in the Laurent expansion:

$$\frac{1}{2\pi i}\int_{\gamma_R} \frac{f(\zeta)\,d\zeta}{\zeta - z} \;=\; \sum_{n\ge 0} \frac{1}{2\pi i}\int_{\gamma_R} \frac{f(\zeta)\,d\zeta}{(\zeta-z_o)^{n+1}}\cdot(z-z_o)^n$$

However, here those integrals are not asserted to have any relation with derivatives of f. The integral over γ_r can be rearranged in a similar way, but now using $|z-z_o| > |\zeta - z_o|$:

$$-\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)\,d\zeta}{\zeta - z} \;=\; -\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)\,d\zeta}{(\zeta-z_o)-(z-z_o)} \;=\; \frac{1}{z-z_o}\cdot\frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)\,d\zeta}{1-\dfrac{\zeta-z_o}{z-z_o}}$$

$$=\; \frac{1}{z-z_o}\cdot\frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)\,\Big(1 + \frac{z-z_o}{\zeta-z_o} + (\frac{z-z_o}{\zeta-z_o})^2 + \ldots\Big)\,d\zeta$$

$$=\; \sum_{n>0} \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)\,d\zeta}{(\zeta-z_o)^{n+1}}\cdot(z-z_o)^n \;=\; \sum_{n<0} \frac{1}{2\pi i}\int_{\gamma_r} \frac{f(\zeta)\,d\zeta}{(\zeta-z_o)^{-n+1}}\cdot(z-z_o)^{-n}$$

This gives the asserted expression for the negative-index part of the Laurent expansion.

For uniqueness, let $\gamma(t) = z_o + \rho\, e^{it}$ be a parametrized circle of radius ρ with $r < \rho < R$, traversed counter-clockwise, and observe that

$$\int_\gamma (\zeta - z_o)^N \, d\zeta \;=\; \int_0^{2\pi} (\rho\, e^{it})^N \, \rho i\, e^{it}\, dt \;=\; i\, \rho^{N+1} \int_0^{2\pi} e^{(N+1)it}\, dt \;=\; \begin{cases} 0 & (\text{for } N \neq -1) \\ 2\pi i & (\text{for } N = -1) \end{cases}$$

An assumption of non-uniqueness of a Laurent expansion of any holomorphic function in the annulus would give a non-trivial Laurent expansion of the identically-zero function. However, integrating both 0 and an alleged Laurent expansion against $(z - z_o)^{N+1}$ shows that the N^{th} coefficient of any Laurent expansion of the zero function on the annulus is 0. This holds for all N. ///

[10.2] Example Laurent coefficients need not be computed from the integral formulas. Often, expanding geometric series gives all one wants. An extreme case is *rational functions*, that is, ratios of polynomials. For example, suppose we want the Laurent expansion of $f(z) = \frac{1}{z-1}$ in the annulus $1 < |z| < \infty$.

$$\frac{1}{z-1} \;=\; \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} \;=\; \frac{1}{z} \cdot \sum_{n=0}^{\infty} \big(\frac{1}{z}\big)^n \;=\; \sum_{n=-\infty}^{n=1} z^n$$

[10.3] Isolated singularities A function f holomorphic on a set of the form $\Omega - \{z_o\}$ where Ω is open and z_o is a point in Ω, is said to have an *isolated singularity* at z_o. The simplest case of this is a *punctured disk*

$$\{z \in \mathbb{C} \;:\; 0 < |z - z_o| < R\}$$

which is an extreme case of an *annulus*, with inner radius 0.

[10.4] Removable singularities A singularity at z_o is *removable* if f extends to a holomorphic function on the whole disk $|z - z_o| < R$.

[10.4.1] Corollary: For f *bounded* near z_o, the isolated singularity at z_o is removable.

Proof: The negative-index Laurent coefficients c_n have bounds

$$\Big|\frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)\, d\zeta}{(\zeta - z_o)^{-n+1}}\Big| \;\leq\; \frac{1}{2\pi} \cdot \frac{\sup_{|\zeta - z_o| = r} |f(\zeta)|}{r^{-n+1}} \cdot (\text{length } \gamma_r) \;=\; \sup_{|\zeta - z_o| = r} |f(\zeta)| \cdot r^n$$

As $r \to 0^+$ the values of f are bounded, so this goes to 0, proving that all the negative-index coefficients are zero. Thus, the Laurent expansion in the punctured disk is actually a (convergent) power series expansion, so is holomorphic, by Abel's theorem. ///

[10.5] Poles When there are only finitely-many non-zero negative-index coefficients, the isolated singularity is a *pole*. When a function f is holomorphic on Ω *except* for a discrete set of points in Ω, at which f has *poles* (as opposed to a worse singularity), f is said to be *meromorphic* in Ω.

[10.5.1] Corollary: At a pole z_o of an otherwise-holomorphic function f,

$$\lim_{z \to z_o} |f(z)| \;=\; +\infty$$

That is, the values of f become large in absolute value. Conversely, if $|f(z)| \to +\infty$ as $z \to z_o$, then the Laurent expansion of f at z_o has finitely-many negative-index coefficients, so the singularity is a pole.

Proof: Let $-N$ be the most-negative index so that the Laurent coefficient is non-zero. As $z \to z_o$, the monomial $(z-z_o)^{-N}$ eventually dominates the rest of the Laurent expansion, and in absolute value goes to $+\infty$.

On the other hand, if $|f(z)| \to +\infty$ at z_o, then $|1/f(z)| \to 0$, so has a removable singularity, and is of the form $\frac{1}{f(z)} = (z-z_o)^N \cdot h(z)$ for h holomorphic and non-vanishing at z_o. Inverting, by the quotient rule $1/h(z)$ is complex-differentiable at z_o, so has a convergent power series expression there. Then $f(z) = (z-z_o)^{-N} \cdot \frac{1}{h(z)}$ gives a Laurent series with finitely-many negative-index coefficients. ///

[10.6] Essential singularities Isolated singularities which are neither removable nor poles are called *essential singularities*. Unlike poles, at which the values of a function become large, at an essential singularity the behavior is more chaotic:

[10.6.1] Corollary: *(Casorati-Weierstrass)* Let z_o be an essential singularity of otherwise-holomorphic f. Then, given $w_1 \in \mathbb{C}$, given $\varepsilon > 0$ and $\delta > 0$, there is z_1 satisfying $|z_1 - z_o| < \delta$ and $|f(z_1) - w_1| < \varepsilon$.

Proof: The idea is to prove a converse: if there is some value w_1 which $f(z)$ stays away from throughout some punctured disk $0 < |z - z_o| < \delta$, then z_o is either removable or a pole. Thus, consider

$$g(z) \;=\; \frac{1}{f(z) - w_1}$$

The hypothesis that f stays away from w_1 assures that the denominator is bounded away from 0, so $g(z)$ is bounded near z_o, so has a *removable* singularity there. Thus, $f(z) = \frac{1}{g(z)} + w_1$. If $g(z_o) \neq 0$, then f has a removable singularity there. If $g(z_o) = 0$, since $g(z)$ is not *identically* 0, then $g(z) = (z-z_o)^N \cdot h(z)$ for some h holomorphic and non-vanishing at z_o. Then $1/h(z)$ is again holomorphic at z_o (by the quotient rule!), so has a convergent power series expansion there. Then $f(z) = w_1 + (z-z_o)^{-N} \frac{1}{h(z)}$ gives the Laurent expansion of f, with finitely-many negative-index terms. ///

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11. *Residues and evaluation of integrals*

The proof of uniqueness of Laurent expansions used the easy but profound fact that

$$\int_\gamma (\zeta - z_o)^N \, d\zeta \;=\; \int_0^{2\pi} (\rho\, e^{it})^N \, \rho i\, e^{it} \, dt \;=\; i\, \rho^{N+1} \int_0^{2\pi} e^{(N+1)it} \, dt \;=\; \begin{cases} 0 & (\text{for } N \neq -1) \\ 2\pi i & (\text{for } N = -1) \end{cases}$$

for γ a circle going counterclockwise around z_o. As in earlier discussions, the same outcome holds for γ *any* reasonable closed curve going once around z_o counterclockwise. This was used above to show that, for f holomorphic on $0 < |z - z_o| < R$ with Laurent expansion

$$f(z) \;=\; \ldots + c_{-2}(z-z_o)^{-2} + c_{-1}(z-z_o)^{-1} + c_0 + c_1(z-z_o) + c_2(z-z_o)^2 + \ldots \qquad (\text{on } 0 < |z-z_o| < R)$$

and γ a small circle around z_o traced counterclockwise,

$$\int_\gamma f \;=\; 2\pi i \cdot c_{-1}$$

It is traditional to name the -1^{th} Laurent coefficient:

$$c_{-1} \;=\; \mathrm{Res}_{z=z_o} f \;=\; \textit{residue} \text{ of } f \text{ at } z_o$$

[11.0.1] Proposition: Let γ be a simple closed curve in a convex open Ω, and f holomorphic on $\Omega - \{z_1, \ldots, z_n\}$ with finitely-many points z_j inside Ω. Then

$$\int_\gamma f \;=\; 2\pi i \cdot \sum_j \mathrm{Res}_{z=z_j} f$$

Proof: The idea is to reduce to the case of small circles γ_j around z_j. Indeed, connecting γ_j to γ by a line segment traversed in to γ_j, then back out to γ, the path integrals of f inbound and outbound *cancel*, so

$$\int_\gamma f \;=\; \sum_j \int_{\gamma_j} f \;=\; \sum_j 2\pi i \mathrm{Res}_{z=z_j} f$$

as claimed. ///

A pointed observation about an important class of examples:

[11.0.2] Proposition: For f be holomorphic at $z = z_o$,

$$\mathrm{Res}_{z=z_o} \frac{f(z)}{z - z_o} \;=\; f(z_o)$$

More generally,

$$\mathrm{Res}_{z=z_o} \frac{f(z)}{(z - z_o)^{n+1}} \;=\; \frac{f^{(n)}(z_o)}{n!}$$

Proof: The power series expansion of f near z_o is $f(z) = f(z_o) + f'(z_o)(z - z_o) + \frac{f''(z_o)}{2!}(z - z_o)^2 + \ldots$, so

$$\frac{f(z)}{z - z_o} \;=\; \frac{f(z_o)}{z - z_o} + f'(z_o) + \frac{f''(z_o)}{2!}(z - z_o) + \ldots$$

showing that the -1^{th} Laurent coefficient is as claimed. The general case is argued similarly. ///

[11.0.3] Example: Many important definite integrals not appearing to refer to complex numbers are amenable to evaluation by residues. The simplest example is the classic

$$I \;=\; \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$$

Of course,

$$I \;=\; \lim_{T \to \infty} \int_{-T}^{T} \frac{dx}{1 + x^2}$$

The trick is to approximate the integral over $[-T, T]$ by a closed-path integral, in this case by the path γ_T that goes from $-T$ to T on the real axis, then traces a semi-circle counter-clockwise in the upper half-plane from T back to $-T$. For $T > 1$, this path encloses just the pole of $f(z) = \frac{1}{1+z^2}$ at $z = i$, so

$$\int_{\gamma_T} \frac{dz}{1 + z^2} \;=\; 2\pi i \cdot \mathrm{Res}_{z=i} \frac{1}{1 + z^2}$$

From the obvious

$$\frac{1}{1 + z^2} \;=\; (z - i)^{-1} \cdot \frac{1}{z + i}$$

with $1/(z+i)$ holomorphic at $z=i$, the proposition just above gives

$$\mathrm{Res}_{z=i}\frac{1}{1+z^2} \;=\; \frac{1}{z+i}\Big|_{z=i} \;=\; \frac{1}{2i}$$

Thus, *by residues*,

$$\int_{\gamma_T}\frac{dz}{1+z^2} \;=\; 2\pi i\cdot\mathrm{Res}_{z=i}\frac{1}{1+z^2} \;=\; 2\pi i\cdot\frac{1}{2i} \;=\; \pi$$

It may seem strange that for $T>1$ the integrals over γ_T do not change as $T\to\infty$, but that is a clear consequence of the behavior of integrals over closed paths.

The next trick is to see that the integral over the semi-circles σ_T of radius T *go to* 0 as $T\to\infty$. The usual crude estimate suffices:

$$\Big|\int_{\sigma_T}\frac{dz}{1+z^2}\Big| \;\le\; (\text{length } \sigma_T)\cdot\max_{z\in\sigma_T}\frac{1}{|1+z^2|} \;=\; \pi T\cdot\frac{1}{T(T-1)} \;\le\; \frac{\pi}{T-1} \;\to\; 0$$

Thus,

$$\int_{-\infty}^{\infty}\frac{dz}{1+z^2} \;=\; \lim_T\Big(\int_{\gamma_T}\frac{dz}{1+z^2}-\int_{\sigma_T}\frac{dz}{1+z^2}\Big) \;=\; \pi-0 \;=\; \pi$$

as we probably already knew for other reasons.

[11.0.4] Example: Let ξ be *real*, and consider[4]

$$\int_{-\infty}^{\infty}\frac{e^{i\xi x}\,dx}{1+x^2} \qquad\qquad (\text{with real } \xi)$$

As in the previous example, where $\xi=0$, we would like to compute this by residues, by looking at integrals from $-T$ to T and then over a semi-circle. Indeed, for $\xi\ge 0$, the exponential is decreasing in size in the upper half-plane, since

$$e^{i\xi(x+iy)} \;=\; e^{i\xi x}\cdot e^{-\xi y}$$

Thus, a nearly identical argument gives

$$\int_{-\infty}^{\infty}\frac{e^{i\xi x}\,dx}{1+x^2} \;=\; 2\pi i\cdot\mathrm{Res}_{z=i}\frac{e^{i\xi z}}{1+z^2} \;=\; 2\pi i\cdot\frac{e^{i\xi z}}{z+i}\Big|_{z=i} \;=\; 2\pi i\cdot\frac{e^{-xi}}{2i} \;=\; \pi e^{-\xi} \qquad (\text{for } \xi\ge 0)$$

However, for $\xi<0$, the exponential blows up in the upper half-plane. Fortunately, the exponential gets smaller in the *lower* half-plane. Thus, we use a semi-circle in the *lower* half-plane. Note that the whole contour is now traced *clockwise*, so there will be a sign:

$$\int_{-\infty}^{\infty}\frac{e^{i\xi x}\,dx}{1+x^2} \;=\; -2\pi i\cdot\mathrm{Res}_{z=-i}\frac{e^{i\xi z}}{1+z^2} \;=\; -2\pi i\cdot\frac{e^{i\xi z}}{z-i}\Big|_{z=-i} \;=\; -2\pi i\cdot\frac{e^{-xi}}{-2i} \;=\; \pi e^{\xi} \qquad (\text{for } \xi<0)$$

Accommodating both signs,

$$\int_{-\infty}^{\infty}\frac{e^{i\xi x}\,dx}{1+x^2} \;=\; \pi e^{-|\xi|}$$

[11.0.5] Remark: It is infeasible to survey all the important examples of integration by residues in the literature in this brief introduction.

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[4] This integral is essentially a normalization of the *Fourier transform* of $1/(1+x^2)$, so has some significance in that context, for example.

12. *Logarithms and complex powers*

A logarithm L of a complex number z should satisfy $e^L = z$. There is inevitable ambiguity: from Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, so $e^{2\pi i} = 1$, and

$$e^L = z \implies e^{L+2\pi i n} = z \qquad \text{(for all } n \in \mathbb{Z})$$

That is, the *imaginary part* of $\log z$ is ambiguous by integer multiples of 2π. The *real part* of the logarithm is unambiguous. Despite the ambiguity, for $z = re^{i\theta}$ with $r \geq 0$ and $\theta \in \mathbb{R}$, we call θ the *argument* of z.

[12.1] Principal branch of logarithm An *unambiguous* holomorphic logarithm can be defined in various *limited* subsets Ω of $\mathbb{C}$, although admittedly *losing* the property $\log(zw) = \log z + \log w$. The property *preserved* is that

$$\frac{d}{dz}\log z = \frac{1}{z}$$

For example, for $z \in \mathbb{C} - (-\infty, 0]$, let γ_z be a line segment connecting 1 to z, by $\gamma_z(t) = tz + (1-t)$, and define the *principal logarithm* $L(z)$ by

$$L(z) = \int_{\gamma_z} \frac{d\zeta}{\zeta} = \int_0^1 \frac{(z-1)\,dt}{tz + (1-t)}$$

The latter integral expression shows that $L(z)$ is holomorphic on $\mathbb{C} - (-\infty, 0]$. Using this integral definition, the property $L(zw) = L(z) + L(w)$ holds *only* for pairs z, w such that the triangle $\Delta(z, w, zw)$ connecting z, w, zw does *not* enclose 0:

$$\int_1^{zw} \frac{d\zeta}{\zeta} = \int_{\frac{1}{z}}^{w} \frac{d(z\zeta)}{(z\zeta)} = \int_{\frac{1}{z}}^{w} \frac{d\zeta}{\zeta} = \int_{\frac{1}{z}}^{1} \frac{d\zeta}{\zeta} + \int_1^w \frac{d\zeta}{\zeta} \qquad \text{(for } \Delta(z,w,zw) \text{ not enclosing } 0)$$

from Cauchy's theorem! Then

$$\int_{\frac{1}{z}}^{1} \frac{d\zeta}{\zeta} + \int_1^w \frac{d\zeta}{\zeta} = \int_1^z \frac{d(\zeta/z)}{(\zeta/z)} + \int_1^w \frac{d\zeta}{\zeta} = \int_1^z \frac{d\zeta}{\zeta} + \int_1^w \frac{d\zeta}{\zeta}$$

showing that $L(zw) = L(z) + L(w)$. When 0 is enclosed, the identity will be *off* by $\pm 2\pi i$, by the residue theorem!

[12.2] Complex powers For z off $(-\infty, 0]$, and for complex α, we can define

$$z^\alpha = e^{L(z)\cdot\alpha} \qquad \text{(principal branch } L(z) \text{ of logarithm)}$$

However, just as with the principal branch of logarithm itself, we can be confident of $(zw)^\alpha = z^\alpha \cdot w^\alpha$ only when the triangle z, w, zw does *not* enclose 0.

13. *The argument principle*

The ambiguity of *argument* and *logarithm* can be put to good use.

[13.0.1] Theorem: *(Argument Principle)* For holomorphic f not identically 0 in a region Ω, and γ a simple closed curve whose interior is inside Ω,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(\zeta)\,d\zeta}{f(\zeta)} = \text{number of zeros of } f \text{ inside } \gamma\text{, counting multiplicities}$$

[13.0.2] Remark: For holomorphic $f(z)$ with power series $c_0 + c_1(z - z_o) + c_2(z - z_o)^2 + \ldots$ with $c_0 = c_1 = \ldots = c_{N-1} = 0$, the *multiplicity* of the zero z_o of f is N. A *simple* zero has multiplicity 1, a *double* zero has multiplicity 2, and so on.

Proof: The function $f'(z)/f(z)$ is holomorphic away from the zeros of f inside γ, so, as usual, we reduce to small circles around the zeros of f, and add up these contributions. With

$$f(z) \;=\; c_N(z-z_o)^N + c_{N+1}(z-z_o)^{N+1} + \ldots$$

with $c_N \neq 0$,

$$f(z) \;=\; c_N(z-z_o)^N\Big(1+\ldots\Big)$$

Also,

$$f'(z) \;=\; Nc_N(z-z_o)^{N-1} + (N+1)c_{N+1}(z-z_o)^N + \ldots \;=\; Nc_N(z-z_o)^{N-1}\cdot\Big(1+\ldots)$$

Thus,

$$\frac{f'(z)}{f(z)} \;=\; = \;\frac{Nc_N(z-z_o)^{N-1}\cdot(1+\ldots)}{c_N(z-z_o)^N\cdot(1+\ldots)} \;=\; \frac{N}{z-z_o}\cdot(1+\ldots) \;=\; \frac{N}{z-z_o} + \text{holomorphic at } z_o$$

Thus, integrating counterclockwise alon a small circle γ_o around z_o,

$$\int_{\gamma_o} \frac{f'(z)\,dz}{f(z)} \;=\; \int_{\gamma_o} \Big(\frac{N}{z-z_o} + \text{holomorphic at } z_o\Big)\;dz \;=\; 2\pi i N + 0$$

Adding these up over all the zeros z_o gives the argument principle formula. ///

[13.0.3] Remark: An important variant of the argument principle comes from $\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z)$. The idea is that traversing a small circle around a zero z_o of order N of f causes the value $f(z)$ to go around zero N times, increasing the *argument* of $f(z)$ by $N\cdot 2\pi$ in the course of returning to the original point.