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Riemann's Explicit/Exact formula

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Even more interesting than a Prime Number Theorem is the *precise* relationship between primes and zeros of zeta found by [Riemann 1859]. A similar idea applies to *any* zeta or L-function with *analytic continuation*, *functional equation*, and *Euler product*.

It took 40 years for [Hadamard 1893], [vonMangoldt 1895], and others to complete Riemann's sketch of the *Explicit Formula* relating primes to zeros of the Euler-Riemann zeta function. Even then, lacking a zero-free strip inside the critical strip, the Explicit Formula does *not* yield a Prime Number Theorem, despite giving a precise relationship between primes and zeros of zeta.

The *idea* is that the equality of the Euler product and Riemann-Hadamard product for zeta allows extraction of an *exact formula* for a suitably-weighted counting of primes, a sum over zeros of zeta, via a contour integration of the logarithmic derivatives. As observed by [Guinand 1947] and [Weil 1952], [Weil 1972], the classical formulas are equalities of values of a certain *distribution*, in the sense of *generalized functions*.

An essential supporting point is *meromorphic continuation* of $\zeta(s)$ via *integral representation(s)* of $\zeta(s)$ in terms of *theta function(s)*. The most symmetrical choice of Schwartz-function [1] data for the theta function gives the *functional equation* of of $\zeta(s)$. This theta function is an example of *automorphic form*. [2] Further, these integral representations give *vertical growth estimates*, critical for invocation of Hadamard's theorem on product expansions of entire functions. This is an archetype: in great generality, automorphic forms are the principal device for proof of non-trivial, essential facts about zeta functions and L-functions.

A key mechanism in analytic continuation and functional equation of $\zeta(s)$ is a functional equation relating two theta series, and showing that the most-symmetrical theta series *is an automorphic form*. This follows from the *Poisson summation formula* for Schwartz functions, itself a corollary of the representability of smooth functions by their *Fourier series*. (See the Supplement.)

The *Gamma function* and its *asymptotics* are needed to use the functional equation of $\zeta(s)$ to obtain the growth bound on $\zeta(s)$ necessary to apply the Hadamard product result. Something simpler than Stirling's formula suffices, admitting a simpler proof. (See the Supplements on asymptotics, and on the functional equation of Gamma.)

[1] Recall that a *Schwartz function* on $\mathbb{R}$ is an infinitely differentiable function all of whose derivatives, including itself, are of *rapid decay* at infinity. That is, $(1+x^2)^\ell \cdot |f^{(k)}(x)|$ is bounded for all k, ℓ.

[2] For practical purposes, *modular form* and *automorphic form* are synonyms, although some sources try to attribute delicately precise meanings.

1. *Riemann's explicit formula*

The dramatic [Riemann 1859] on the relation between primes and zeros of the zeta function depended on many ideas undeveloped in Riemann's time. Thus, the following sketch, very roughly following Riemann, is far from a proof, but uncovers supporting ideas *needed* to produce a proof, and shows the reward for doing following up. [3]

Riemann knew from Euler that $\zeta(s)$ has an *Euler product* expansion in a half-plane

$$\zeta(s) \;=\; \sum_{n\geq 1} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \frac{1}{1-\frac{1}{p^s}} \qquad\qquad (\text{for } \mathrm{Re}\, s > 1)$$

As we discuss just below, [Riemann 1859] proved that $\zeta(s)$ has a *meromorphic continuation* so that $(s-1)\zeta(s)$ is *entire*, with $0 = \zeta(0) = \zeta(-2) = \zeta(-4) = \ldots$. [4] The negative even integer are the *trivial* zeros of $\zeta(s)$. Riemann's first inspiration was to imagine that $\zeta(s)$ has a *product expansion* in terms of its zeros [5]

$$(s-1)\,\zeta(s) \;=\; e^{a+bs} \prod_\rho \Big(1-\frac{s}{\rho}\Big)\, e^{s/\rho} \cdot \prod_{n=1}^\infty \Big(1+\frac{s}{2n}\Big)\, e^{-s/2n} \qquad (\rho \text{ non-trivial zero of } \zeta, \text{ for all } s\in\mathbb{C})$$

[Hadamard 1893] proved this. *Then*, more tangible information can be extracted from the equality of the two products

$$(s-1) \prod_p \frac{1}{1-\frac{1}{p^s}} \quad=\quad e^{a+bs} \prod_\rho \Big(1-\frac{s}{\rho}\Big)\, e^{s/\rho} \cdot \prod_{n=1}^\infty \Big(1+\frac{s}{2n}\Big)\, e^{-s/2n} \qquad (\mathrm{Re}\, s > 1)$$

Taking logarithmic derivatives of both sides, using $-\log(1-x) = x + x^2/2 + x^3/3 + \ldots$ on the left-hand side:

$$\frac{1}{s-1} - \sum_{m\geq 1,\; p} \frac{\log p}{p^{ms}} \;=\; b \;+\; \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \;+\; \sum_n \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big)$$

A slight rearrangement:

$$\sum_{m\geq 1,\; p} \frac{\log p}{p^{ms}} \;=\; \frac{1}{s-1} - b \;-\; \sum_\rho \Big(\frac{1}{s-\rho} + \frac{1}{\rho}\Big) \;-\; \sum_n \Big(\frac{1}{s+2n} - \frac{1}{2n}\Big) \qquad (\text{for } \mathrm{Re}\, s > 1)$$

[1.0.1] Remark: The two sides of the equality of logarithmic derivatives are very different. The logarithmic derivative of the Euler product converges well, but only in right half-planes. The logarithmic derivative of

[3] Riemann's paper was remarkable: he intuited an interesting objective, and sketched the necessary supporting ideas without waiting for corroboration. Thus, he was able to see the interest of the *conclusion*, giving a powerful motivation to investigation of the supporting technical ideas.

[4] It is absolutely not obvious that $\zeta(s)$ vanishes at negative even integers. This will follow from the *functional equation*. Even so, Euler had already done computations that could be interpreted as proving this!

[5] This product expansion idea did not occur in a complete vacuum. First, of course, for non-zero complex $\alpha_1, \ldots, \alpha_n$, $(1-\frac{z}{\alpha_1})\ldots(1-\frac{z}{\alpha_n})$ is the unique polynomial of the form $1+\ldots$ with the indicated zeros. Euler's evaluation of $\sum_n \frac{1}{n^2}$ by imagining (and later proving) $\sin \pi z = \pi z \prod_n (1-\frac{z^2}{n^2})$ was well known. Also, Euler's product expansion of the inverse of the Gamma-function $\Gamma(s) = \int_0^\infty t^s\, e^{-t}\, \frac{dt}{t}$ as $\frac{1}{\Gamma(s)} = z e^{\gamma z} \prod_n (1+\frac{z}{n})\, e^{-z/n}$ was well known to Riemann. But the zeta function $\zeta(s)$ is much less elementary.

the Riemann-Hadamard product does *not* converge strongly, but is *not* restricted to a half-plane, and its poles are exhibited explicitly by the expression.

Again diverging slightly from Riemann's original treatment, we intend to apply the Perron identity [6] (see Appendix)

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{Y^s}{s}\,ds \;=\; \lim_{T\to\infty}\frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT}\frac{Y^s}{s}\,ds \;=\; \begin{cases} 1 & (\text{for } Y>1) \\ 0 & (\text{for } 0<Y<1)\end{cases} \qquad (\text{for } \sigma>0)$$

to the log-derivative identity multiplied by X^s/s. Assuming we can apply the Perron identity *term-wise* to X^s/s times the left-hand side, this would give

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s}{s}\sum_{m,p}\frac{\log p}{p^{ms}}\,ds \;=\; \sum_{m,p}\log p\cdot\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s\cdot p^{-ms}}{s}\,ds \;=\; \sum_{p^m<X}\log p$$

Assuming we can use *residues* term-wise to evaluate X^s/s times the right-hand side, with $\sigma>1$, this would give

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{X^s}{s}\cdot\Big(\frac{1}{s-1}-b-\sum_\rho\Big(\frac{1}{s-\rho}+\frac{1}{\rho}\Big)-\sum_n\Big(\frac{1}{s+2n}-\frac{1}{2n}\Big)\Big)\,ds$$

$$=\;(X-1)-b-\sum_\rho\Big(\frac{X^\rho}{\rho}+\frac{1}{-\rho}+\frac{1}{\rho}\Big)-\sum_n\Big(\frac{X^{-2n}}{-2n}+\frac{1}{2n}-\frac{1}{2n}\Big)\;=\;X-(b+1)-\sum_\rho\frac{X^\rho}{\rho}+\sum_{n\ge1}\frac{X^{-2n}}{2n}$$

Thus, we would have [vonMangoldt 1893]'s reformulation of **Riemann's Explicit Formula:**

$$\sum_{p^m<X}\log p \;=\; X-(b+1)-\sum_\rho\frac{X^\rho}{\rho}+\sum_{n\ge1}\frac{X^{-2n}}{2n}$$

Slightly more precisely, because of the way the Perron integral transform is applied, and the fragility of the convergence, we should say

$$\sum_{p^m<X}\log p \;=\; X-(b+1)-\lim_{T\to\infty}\sum_{|\mathrm{Im}(\rho)|<T}\frac{X^\rho}{\rho}+\sum_{n\ge1}\frac{X^{-2n}}{2n}$$

[1.0.2] Remark: As in Riemann's original, the above sketch has many potential gaps, despite clear intentions. The existence and convergence of the Hadamard product needs both *generalities* about Weierstraß-Hadamard product expressions for entire functions of prescribed growth, grounded in basic complex analysis, and *specifics* about the growth of the *analytic continuation* of $\zeta(s)$. The analytic continuation of $\zeta(s)$ is discussed in the next section, and growth properties later. The growth properties depend on Stirling-Laplace asymptotics of the Gamma function $\Gamma(s)$, and the *Phragmén-Lindelöf* theorem. Background on these topics in complex analysis is recalled in supplements.

[1.1] Non-trivial zeros ρ of $\zeta(s)$

The convergent Euler product shows that $\zeta(s)\neq 0$ in the half-plane $\mathrm{Re}(s)>1$. Subsequent considerations [7] show that the only possible non-trivial zeros are in the *critical*

[6] Perron's identity is completely standard by now, but was not part of Riemann's approach. Invocation of the Perron identity allows a somewhat simpler approach than Riemann's original, due to von Mangoldt and others.

[7] The analytic continuation and functional equation (below), and relatively elementary properties of the Gamma function $\Gamma(s)=\int_0^\infty t^s\,e^{-t}\,\frac{dt}{t}$

strip $0 \le \mathrm{Re}(s) \le 1$. In 1896, Hadamard and de la Vallée-Poussin independently proved that there are no zeros on the edges $\mathrm{Re}(s) = 0, 1$ of the critical strip, which they used to prove the *Prime Number Theorem*.

The functional equation shows that if ρ is a non-trivial zero, then $1 - \rho$ is a non-trivial zero. The property $\zeta(\bar{s}) = \overline{\zeta(s)}$ shows that if ρ is a non-trivial zero, then $\bar{\rho}$ is a non-trivial zero.

[1.2] The Riemann Hypothesis

After the main term X in the right-hand side of the explicit formula, the next-largest terms would be the X^ρ/ρ summands, with $0 \le \mathrm{Re}(\rho) \le 1$ due to the Euler product and functional equation. The *Riemann Hypothesis* is that all the non-trivial zeros ρ have $\mathrm{Re}(\rho) = \frac{1}{2}$. With a bound like $T \log T$ on the number of zeros below height T, the Riemann hypothesis is equivalent to an error term of order $X^{\frac{1}{2}+\varepsilon}$ in the Prime Number Theorem, for all $\varepsilon > 0$.

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2. *Analytic continuation and functional equation of* $\zeta(s)$

The following ideas gained publicity and importance from Riemann, but were apparently known to some degree before Riemann.

The key is that the completed zeta function has an *integral representation* in terms of an *automorphic form*, the simplest *theta function*. Both the *analytic continuation* and the *functional equation* of zeta follow from this integral representation using a parallel functional equation of the theta function, the latter demonstrated by *Poisson summation*, which comes from *Fourier series*.

Modernizing this idea, the analytic continuation can be separated from the functional equation, as in the following.

[2.1] Elementary-but-insufficiently-enlightening argument for analytic continuation

Simple calculus can extend the domain of $\zeta(s)$ as far to the left as we want. The idea is to pay attention to *quantitative* aspects of the integral test. First, by comparison to $\int_1^\infty \frac{dx}{x^s}$, the sum $\zeta(s) = \sum_1^\infty \frac{1}{n^s}$ converges for $\mathrm{Re}(s) > 1$.

To push this further, it is standard to proceed as follows.

$$\zeta(s) - \frac{1}{s-1} \;=\; \zeta(s) - \int_1^\infty \frac{dx}{x^s} \;=\; \sum_n \Big(\frac{1}{n^s} - \int_n^{n+1} \frac{dx}{x^s}\Big) \;=\; \sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big)$$

Even for complex s, we have a Taylor-Maclaurin expansion with *error term*[8]

$$(n+1)^{1-s} \;=\; \big(n \cdot (1 + \frac{1}{n})\big)^{1-s} \;=\; n^{1-s} \cdot \big(1 + \frac{1-s}{n} + O(\tfrac{1}{n^2})\big) \;=\; \frac{1}{n^{s-1}} - \frac{s-1}{n^s} + O(\frac{s-1}{n^{s+1}})$$

The constant in the big-O term is *uniform* in n for fixed s. Thus,

$$\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big] \;=\; \frac{1}{n^s} - \frac{1}{n^s} + \frac{1}{s-1} O\big(\frac{1}{n^{s+1}}\big) \;=\; O\big(\frac{1}{n^{s+1}}\big)$$

That is, for fixed[9] $\mathrm{Re}(s) > 0$, we have *absolute convergence* of

$$\sum_n \Big(\frac{1}{n^s} - \frac{1}{s-1}\Big[\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\Big]\Big) \qquad \text{(for } \mathrm{Re}(s) > 0\text{)}$$

[8] Landau's *big-O* notation $f(x) = g(x) + O(h(x))$ means that, as $x \to \infty$ (or some other limit), $|f(x) - g(x)|$ is bounded by a constant multiple of $|h(x)|$.

[9] In fact, the big-O constant is also *uniform* for s in compacts inside $\mathrm{Re}(s) > 0$. Thus, the series converges *locally uniformly on compacts*, so does give a *holomorphic* function.

in the larger region $\mathrm{Re}(s) > 0$.

[2.1.1] Remark: Iterating the idea of approximating sums by integrals gives a comparable extension to $\mathrm{Re}(s) > -\ell$ for all ℓ, as Euler already effectively found, systematically by *Euler-Maclaurin summation.* However, such continuations give no clues about functional equations, and certainly not about Riemann's explicit formula.

[2.2] Slight modernization of Riemann's argument We update Riemann's idea to avoid needless artifacts. Both the original and this update are archetypes. [10] Let $f(x)$ be *any* very well-behaved function on $\mathbb{R}$, that is, infinitely differentiable, and it and all its derivatives are rapidly decreasing at infinity. These are *Schwartz functions*, after [Schwartz 1950/51]. Further, take f *even*, that is $f(-x) = f(x)$. The even Schwartz function f is a *dummy*, insofar as only its general properties are used. In effect, Riemann's choice was the Gaussian $f(x) = e^{-\pi x^2}$, based on connections to Jacobi's theta functions, as we see along the way. A *theta function*[11] associated to the even Schwartz function f is

$$\theta_f(y) \;=\; \sum_{n\in\mathbb{Z}} f(y\cdot n) \qquad\qquad (\text{for } y > 0)$$

and associated *Gamma function*[12]

$$\Gamma_f(s) \;=\; \int_0^\infty t^s\, f(t)\, \frac{dt}{t}$$

First, we have the *integral representation*, from which will follow the meromorphic continuation and functional equation:

[2.2.1] Proposition: $\displaystyle\int_0^\infty y^s\, \frac{\theta_f(y) - f(0)}{2}\, \frac{dy}{y} \;=\; \Gamma_f(s)\cdot\zeta(s) \qquad\qquad (\text{for } \mathrm{Re}(s) > 1)$

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[10] Riemann's original line of argument was brought to completion by [Hecke 1918/20]. Substantial modernization occurred in [Matchett 1946], [Iwasawa 1950/52], [Iwasawa 1952], and [Tate 1950/1967]. In particular, these sources observed that certain details involving *theta functions* were less essential than previously believed. Nevertheless, the *automorphic* nature of theta functions was *also* important in its own right.

[11] Again, Riemann used $f(u) = e^{-\pi u^2}$, and, consistent with an existing convention at the time, in effect defined

$$\theta(iy) \;=\; \sum_{n\in\mathbb{Z}} f(\sqrt{y}\cdot n) \qquad\qquad (\text{with Gaussian } f(u) = e^{-\pi u^2})$$

That is, the argument of θ is iy rather than y, and $\sqrt{y}$ enters on the right side, rather than y. Further, the Gaussian extends to an *entire* function, and this theta function extends to a holomorphic function, the simplest *Jacobi theta function*, on the upper half-plane $\mathfrak{H}$:

$$\theta(z) \;=\; \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} \qquad\qquad (\text{with } z \in \mathfrak{H})$$

[12] With Gaussian $f(x) = e^{-\pi x^2}$, this construction gives an exponential multiple of the standard Gamma function at $\frac{s}{2}$:

$$\Gamma_f(s) \;=\; \int_0^\infty t^s\, e^{-\pi x^2}\, \frac{dx}{x} \;=\; \tfrac{1}{2}\int_0^\infty t^{\frac{s}{2}}\, e^{-\pi x}\, \frac{dx}{x} \;=\; \tfrac{1}{2}\,\pi^{-\frac{s}{2}}\int_0^\infty t^{\frac{s}{2}}\, e^{-x}\, \frac{dx}{x} \;=\; \tfrac{1}{2}\,\pi^{-\frac{s}{2}}\,\Gamma(\tfrac{s}{2})$$

Proof: The $n = 0$ (constant) term $f(0)$ of $\theta_f(y)$ is the only summand not rapidly decreasing. The even-ness of f makes the $\pm n$ terms have equal contributions to $\theta_f(y)$. Thus, interchanging sum and integral, and replacing y by y/n,

$$\int_0^\infty y^s \, \frac{\theta_f(y) - f(0)}{2} \, \frac{dy}{y} \;=\; \sum_{n\geq 1} \int_0^\infty y^s \, f(yn) \, \frac{dy}{y} \;=\; \sum_{n\geq 1} n^{-s} \int_0^\infty y^s \, f(y) \, \frac{dy}{y} \;=\; \sum_{n\geq 1} n^{-s} \, \Gamma_f(s)$$

as claimed. ///

[2.2.2] Remark: The measure $\frac{dy}{y}$ is the natural multiplication-invariant measure on the positive reals.

[2.2.3] Theorem: The completed zeta function $\Gamma_f(s) \cdot \zeta(s)$ has a meromorphic continuation to $s \in \mathbb{C}$, and $s(s-1) \cdot \Gamma_f(s) \cdot \zeta(s)$ is *entire.*

[2.2.4] Remark: Repeated integration by parts shows that $\Gamma_f(s)$ itself has a meromorphic continuation:

$$\Gamma_f(s) \;=\; \int_0^\infty t^s \, f(t) \, \frac{dt}{t} = \int_0^\infty \frac{t^{s+1}}{s} \, f'(t) \, \frac{dt}{t} = \int_0^\infty \frac{t^{s+2}}{s(s+1)} \, f''(t) \, \frac{dt}{t} = \int_0^\infty \frac{t^{s+3}}{s(s+1)(s+2)} \, f'''(t) \, \frac{dt}{t} = \ldots$$

Since all the derivatives of f are of rapid decay, these expressions give an extension of $\Gamma_f(s)$ to $s \in \mathbb{C}$ except for at worst $s = 0, -1, -2, -3, \ldots$.

Proof: Break the integral of the integral representation into two parts:

$$\Gamma_f(s) \cdot \zeta(s) \;=\; \int_1^\infty y^s \, \frac{\theta_f(y) - f(0)}{2} \, \frac{dy}{y} \;+\; \int_0^1 y^s \, \frac{\theta_f(y) - f(0)}{2} \, \frac{dy}{y}$$

It is not hard to check that $\frac{\theta_f(y) - \theta_f(0)}{2}$ is rapidly decreasing at $+\infty$, so the integral on $[1, \infty)$ is absolutely convergent (and uniformly for s in compacts) for all $s \in \mathbb{C}$.

The behavior of $\theta_f(y)$ as $y \to 0^+$ is harder to analyze, and is best done by the following device.

The trick is to convert the integral on $[0, 1]$ to an integral over $[1, \infty)$, up to two elementary terms. The new integral over $[1, \infty)$ will involve the theta function $\theta_{\widehat{f}}$ attached to the *Fourier transform*

$$\widehat{f}(x) \;=\; \int_{\mathbb{R}} e^{-2\pi i x \xi} \, f(\xi) \, d\xi$$

of f. We grant for the moment that Fourier transform maps the Schwartz space to itself, as is directly verifiable in concrete examples such as the Gaussian $f(x) = e^{-\pi x^2}$. Simply by changing variables in the integral, we recall a homogeneity property of the Fourier transform:

$$\widehat{f}(x/y) \;=\; \int_{\mathbb{R}} e^{-2\pi i \frac{x}{y} \xi} \, f(\xi) \, d\xi \;=\; |y| \int_{\mathbb{R}} e^{-2\pi i x \xi} \, f(y\xi) \, d\xi \;=\; |y| \cdot (f \circ y)\widehat{\ }(x)$$

by replacing ξ by ξy in the integral, where $(f \circ y)(\xi) = f(y\xi)$. We grant ourselves the standard *Poisson summation formula*

$$\sum_{n \in \mathbb{Z}} F(n) \;=\; \sum_{n \in \mathbb{Z}} \widehat{F}(n) \qquad \text{(for Schwartz functions } F\text{)}$$

(See the Supplement for proof.) Letting $F(x) = f(yx)$ and using the homogeneity property of Fourier transform, this is

$$\sum_{n \in \mathbb{Z}} f(y \cdot n) \;=\; \sum_{n \in \mathbb{Z}} \frac{1}{y} \, \widehat{f}\Big(\frac{1}{y} \cdot n\Big) \qquad \text{(for } y > 0\text{)}$$

Thus,

$$\theta_f(y) \;=\; \sum_{n\in\mathbb{Z}} f(yn) \;=\; \sum_{n\in\mathbb{Z}} \frac{1}{y}\widehat{f}(n) \;=\; \frac{1}{y}\cdot\theta_{\widehat{f}}\Big(\frac{1}{y}\Big)$$

This gives a way to flip the interval $[0,1]$ to $[1,\infty)$, by replacing y by $1/y$, accommodating the anomalous terms for $n=0$ separately:

$$\int_0^1 y^s\,\frac{\theta_f(y)-f(0)}{2}\,\frac{dy}{y} \;=\; \int_0^1 y^s\,\frac{\frac{1}{y}\theta_{\widehat{f}}(\frac{1}{y})-f(0)}{2}\,\frac{dy}{y} \;=\; \int_0^1 y^s\,\frac{\frac{1}{y}\theta_{\widehat{f}}(\frac{1}{y})-\frac{1}{y}\widehat{f}(0)}{2}+\frac{\frac{1}{y}\widehat{f}(0)-f(0)}{2}\,\frac{dy}{y}$$

$$= \int_1^\infty y^{-s}\,\frac{y\theta_{\widehat{f}}(y)-y\widehat{f}(0)}{2}+\int_0^1 y^s\,\frac{\frac{1}{y}\widehat{f}(0)-f(0)}{2}\,\frac{dy}{y}$$

$$= \int_1^\infty y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\,\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\int_0^1 y^{s-1}\,\frac{dy}{y} \;-\; \frac{f(0)}{2}\int_0^1 y^s\,\frac{dy}{y}$$

$$= \int_1^\infty y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\,\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\,\frac{1}{s-1} \;-\; \frac{f(0)}{2}\,\frac{1}{s}$$

The integral on $[1,\infty)$ is entire in s, since $\theta_{\widehat{f}}(y)-\widehat{f}(0)$ is rapidly decreasing at ∞. The two elementary terms have obvious meromorphic continuations. Thus,

$$\Gamma_f(s)\cdot\zeta(s) \;=\; \int_1^\infty \Big(y^s\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}+y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\Big)\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\,\frac{1}{s-1} \;-\; \frac{f(0)}{2}\,\frac{1}{s}$$

Again, the integral is *entire*, and the elementary terms give the only poles, which are at $s=0,1$. ///

[2.2.5] Remark: The expression

$$\Gamma_f(s)\cdot\zeta(s) \;=\; \int_1^\infty \Big(y^s\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}+y^{1-s}\,\frac{\theta_{\widehat{f}}(y)-\widehat{f}(0)}{2}\Big)\frac{dy}{y} \;+\; \frac{\widehat{f}(0)}{2}\,\frac{1}{s-1} \;-\; \frac{f(0)}{2}\,\frac{1}{s}$$

gives a bit more information than the bare statement of the theorem, namely, it tells the residues of the poles at $s=0,1$, and shows a certain potential symmetry, as in the following.

For f with $\widehat{f}=f$ Riemann's original symmetrical result is recovered:

[2.2.6] Theorem: *(Riemann)* The *completed* zeta function

$$\xi(s) \;=\; \pi^{-\frac{s}{2}}\,\Gamma(\frac{s}{2})\,\zeta(s)$$

has an analytic continuation to $s\in\mathbb{C}$, except for simple poles at $s=0,1$, and has the *functional equation*

$$\xi(1-s) \;=\; \xi(s)$$

Proof: Various means show that $f(x)=e^{-\pi x^2}$ is its own Fourier transform. Thus, the expression in the proof of the previous theorem becomes symmetrical in $s\leftrightarrow 1-s$, and the artifact of the coefficient of $\frac{1}{2}$ on both sides can be discarded. ///

[2.2.7] Remark: The leading factor $\pi^{-s/2}\Gamma(\frac{s}{2})$ should *not* be construed as objectionable in any way, but, rather, as something that really does *belong* with $\zeta(s)$. The $\pi^{-s/2}\Gamma(\frac{s}{2})$ is called the **gamma factor** for $\zeta(s)$.

In the context of the *Euler product* the modern viewpoint is that the gamma factor is a further Euler factor corresponding to the *prime*[13] ∞.

3. *Appendix: Perron identity*

These contour-integral identities extract information from spectral identities and function-theoretic identities. *One* spectral identity is transformed into *another*, by a Fourier transform. Choices are made to heighten an *asymmetry*, wherein one side is seemingly elementary, and the other is whatever it must be.

[3.1] Heuristic The best-known identity starts from the *idea* that for $\sigma > 0$

$$\int_{\sigma-i\infty}^{\sigma-i\infty} \frac{X^s}{s}\, ds \;=\; \begin{cases} 1 & (\text{for } X > 1) \\ \\ 0 & (\text{for } 0 < X < 1) \end{cases} \qquad\qquad (\text{convergence?})$$

The *idea* of the proof of this identity is that, for $X > 1$, the contour of integration slides indefinitely to the left, eventually vanishing, picking up the residue at $s = 0$, while for $0 < X < 1$, the countour slides indefinitely to the right, eventually vanishing, picking up *no* residues.

The *idea* of the application is that this identity can extract *counting* information from a meromorphic continuation of a Dirichlet series: for example, from

$$\sum_n \frac{a_n}{n^s} \;=\; f(s) \qquad\qquad (\text{left-hand side convergent for } \operatorname{Re} s > 1)$$

we would have

$$\sum_{n<X} a_n \;=\; \text{sum of residues of } X^s\, f(s)/s$$

That is, the *counting* function $\sum_{n<X} a_n$ is *extracted* from the analytic object $\sum_\lambda a_n/n^s$ by the contour integration. With f a logarithmic derivative, such as $f(s) = \zeta'(s)/\zeta(s)$, the poles of f are mostly the zeros of ζ.

However, the tails of these integrals are fragile.

[3.2] Simple precise assertion The elegant simplicity of the idea about moving lines of integration must be elaborated for correctness: for fixed $\sigma > 0$, for $T > 0$, we claim that

$$\int_{\sigma-iT}^{\sigma-iT} \frac{X^s}{s}\, ds \;=\; \begin{cases} 1 + O_\sigma(\frac{X^\sigma}{T\cdot \log X}) & (\text{for } X > 1) \\ \\ O_\sigma(\frac{X^\sigma}{T\cdot |\log X|}) & (\text{for } 0 < X < 1) \end{cases}$$

The proof is a precise form of the idea of sliding vertical contours. That is, for $X > 1$, consider the contour integral around the rectangle with *right* edge $\sigma \pm iT$, namely, with vertices $\sigma - iT$, $\sigma + it$, $-B + iT$, $-B - iT$, with $B \to +\infty$. For $0 < X < 1$ consider the contour integral around the rectangle with *left* edge $\sigma \pm iT$, namely, with vertices $\sigma - iT$, $\sigma + it$, $B + iT$, $B - iT$, with $B \to +\infty$.

[13] An insight of modern times is that the completion $\mathbb{R}$ should whenever possible be put on an even footing with the other p-adic completions $\mathbb{Q}_p$ of $\mathbb{Q}$. Thus, although there is no actual prime ∞ in $\mathbb{Z}$ (or anywhere else), the objects that accompany genuine primes p and completions $\mathbb{Q}_p$ often have analogues for $\mathbb{R}$, so we *backform* to refer to the *prime* ∞. One attempt to be less bold in this regard is to speak of *places* rather than *primes*, but there's little point in fretting about this.

For both $X > 1$ and $0 < X < 1$, the $\pm(B \pm iT)$ edge of the rectangle is dominated by

$$\int_{-T}^{T} \frac{e^{-B|\log X|}}{|B \pm it|}\,dt \;\ll\; T \cdot \frac{e^{-B|\log X|}}{B} \;\to\; 0 \qquad\qquad (\text{as } B \to +\infty)$$

in both cases, the top and bottom edges of the rectangle are dominated by

$$X^\sigma \cdot \int_0^\infty \frac{e^{-u|\log X|}}{|(\sigma \pm u) + iT|}\,du \;\ll\; X^\sigma \cdot \int_0^\infty \frac{e^{-u|\log X|}}{T}\,du \;\ll\; \frac{X^\sigma}{T \cdot |\log X|}$$

This proves the claim. Replacing X by e^X in the estimate gives the equivalent

$$\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma - iT} \frac{e^{sX}}{s}\,ds \;=\; \begin{cases} 1 + O_\sigma(\frac{e^{\sigma X}}{T \cdot X}) & (\text{for } X > 0) \\ O_\sigma(\frac{e^{\sigma X}}{T \cdot |X|}) & (\text{for } X < 0) \end{cases}$$

[3.3] Hazards When the quantity X above is summed, especially if the summation is over a set whose precise specifications are difficult, the denominators of the big-O error terms may blow up. In situations such as

$$\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma + iT} \Big(\sum_j a_j\, e^{-sX_j}\Big) \frac{e^{sX}}{s}\,ds \;=\; \sum_{j\,:\,X_j < X} a_j \;+\; \sum_j a_j \cdot O_\sigma\Big(\frac{e^{\sigma(X - X_j)}}{T \cdot |X - X_j|}\Big)$$

the distribution of the values X_j has an obvious effect on the convergence of the error term.

[3.4] The other side of the equation A desired and plausible conclusion such as

$$\lim_T \frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma - iT} f(s)\,\frac{e^{sX}}{s}\,ds \;=\; \big(\text{sum of } \mathrm{Res}_{s=\rho}\, f(s) \cdot \tfrac{e^{\rho X}}{\rho}\big)$$

summed over poles ρ of f in the left half-plane $\mathrm{Re}\, s < \sigma$, requires that the contour integrals over the other three sides of the rectangle with side $\sigma \pm iT$ go to 0, and that the tails of the vertical integral go to 0. The integral over the large rectangle will be evaluated with X large positive, so the decay condition applies to f to the *left*. The left side of the rectangle will go to 0 for large enough positive X when $f(s)$ has at worst exponential growth to the left, that is, when $f(s) \ll e^{-C \cdot |\mathrm{Re}\, s|}$ for *some* large-enough C and $\mathrm{Re}\, s \to -\infty$. The top and bottom are more fragile, since e^{sX}/s does not have strong decay vertically.

Not unexpectedly, the *poles* of f near $\sigma + iT$ may *bunch up* as T grows, so that a countour integral must be **threaded** between them, and the corresponding integral will be somewhat larger simply because of proximity to these poles. This contribution to vertical growth of f is significant in examples, and motivates alternatives.

[3.5] Variant identities When e^{sX}/s is altered to help convergence of the integral against the *counting* aspect is inevitably altered. The proofs of variants follow the same straightforward line as above for the simplest case. Rather than replacing e^{sX}/s with e^{sX}/s^2, a better effect is achieved with $e^{sX}/s(s+1)$. In fact, for $\theta > 0$ and $1 \le \ell \in \mathbb{Z}$

$$\frac{1}{2\pi i}\int_{\sigma - iT}^{\sigma - iT} \frac{e^{sX}}{s(s+\theta)(s+2\theta)\dots(s+\ell\theta)}\,ds \;=\; \begin{cases} \frac{1}{\ell!\theta^\ell}(1 - e^{-\theta X})^\ell + O_\sigma(\frac{e^{\sigma X}}{T^2 \cdot X}) & (\text{for } X > 0) \\ O_\sigma(\frac{e^{\sigma X}}{T^2 \cdot |X|}) & (\text{for } X < 0) \end{cases}$$

Indeed, the residues at the poles $0, -\theta, -2\theta, \dots, -\ell\theta$ sum to

$$\frac{e^{0\cdot X}}{(0+\theta)(0+2\theta)\cdots(0+(\ell-1)\theta)(0+\ell\theta)} + \frac{e^{-\theta\cdot X}}{(-\theta+0)(-2\theta+\theta)\cdots(-\theta+(\ell-1)\theta)(-\theta+\ell\theta)}$$
$$+ \frac{e^{-2\theta\cdot X}}{(-2\theta+0)(-2\theta+\theta)\cdots(-2\theta+\ell\theta)} + \ldots + \frac{e^{-\ell\theta\cdot X}}{(-\ell\theta+0)(-\ell\theta+\theta)\cdots(-\ell\theta+(\ell-1)\theta)}$$
$$= \frac{1}{\ell!\,\theta^\ell} - \frac{e^{-\theta X}}{1!\,(\ell-1)!\,\theta^\ell} + \frac{e^{-2\theta X}}{2!\,(\ell-2)!\,\theta^\ell} + \ldots \pm \frac{e^{-\ell\theta X}}{\ell!\,0!\,\theta^\ell} \quad = \quad \frac{(1-e^{-\theta X})^\ell}{\ell!\,\theta^\ell}$$

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