Weierstrass and Hadamard products

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Apart from factorization of polynomials, perhaps the oldest product expression is Euler’s

\[
\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)
\]

Granting this, Euler equated the power series coefficients of \(z^2\), evaluating \(\zeta(2)\) for the first time:

\[
\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}
\]

The \(\Gamma\)-function factors:

\[
\int_{0}^{\infty} e^{-t} t^{z} \frac{dt}{t} = \Gamma(z) = \frac{1}{z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)e^{-z/n}}
\]

where the Euler-Mascheroni constant \(\gamma\) is essentially defined by this relation. The integral (Euler’s) converges for \(\text{Re}(z) > 0\), while the product (Weierstrass’) converges for all complex \(z\) except non-positive integers. Granting this, the \(\Gamma\)-function is visibly related to \(\text{sine}\) by

\[
\frac{1}{\Gamma(z) \cdot \Gamma(-z)} = -z^2 \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = -\frac{z}{\pi} \cdot \sin \pi z
\]

because the exponential factors are linear, and can cancel.

Linear exponential factors are exploited in Riemann’s explicit formula [Riemann 1859], derived from equality of the Euler product and Hadamard product [Hadamard 1893] for the zeta function \(\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}\) for \(\text{Re}(s) > 1\):

\[
\prod_{p \text{ prime}} \frac{1}{1-p^{-s}} = \zeta(s) = \frac{e^{a+bs}}{s-1} \cdot \prod_{\rho} \left(1 - \frac{s}{\rho}\right)e^{\rho s} \cdot \prod_{n=1}^{\infty} \left(1 + \frac{s}{2n}\right)e^{-s/2n}
\]

where the product expansion of \(\Gamma(\frac{s}{2})\) is visible, corresponding to trivial zeros of \(\zeta(s)\) at negative even integers, and \(\rho\) ranges over all other, non-trivial zeros, known to be in the critical strip \(0 < \text{Re}(s) < 1\).

The hard part of the proof (below) of Hadamard’s theorem is adapted from [Ahlfors 1953/1966], with various rearrangements. A somewhat different argument is in [Lang 1993]. We recall some standard (folkloric?) proofs of supporting facts about harmonic functions, starting from scratch.
1. Weierstrass products

Given a sequence of complex numbers \( z_j \) with no accumulation point in \( \mathbb{C} \), we will construct an entire function with zeros exactly the \( z_j \). This is essentially elementary.

[1.1] Basic construction

Taylor-MacLaurin polynomials of \(-\log(1 - z)\) will play a role: let

\[
p_n(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \ldots + \frac{z^n}{n}
\]

We will show that there is a sequence of integers \( n_j \) giving an absolutely convergent infinite product vanishing precisely at the \( z_j \), with vanishing at \( z = 0 \) accommodated by a suitable leading factor \( z^{m_j} \):

\[
z^m \prod_j \left( 1 - \frac{z}{z_j} \right) e^{p_{n_j}(z/z_j)} = z^m \prod_j \left( 1 - \frac{z}{z_j} \right) \exp \left( \frac{z}{z_j} + \frac{z^2}{2z_j} + \frac{z^3}{3z_j} + \ldots + \frac{z^{n_j}}{n_jz_j} \right)
\]

Absolute convergence of \( \sum_j \log(1 + a_j) \) implies absolute convergence of the infinite product \( \prod_j (1 + a_j) \). Thus, we show that

\[
\sum_j \log \left( 1 - \frac{z}{z_j} \right) + p_{n_j} \left( \frac{z}{z_j} \right) < \infty
\]

Fix a large radius \( R \), keep \( |z| < R \), and ignore the finitely-many \( z_j \) with \( |z_j| < 2R \), so in the following we have \( |z/z_j| < \frac{1}{2} \). Using the power series expansion of \( \log \),

\[
\left| \log(1 - \frac{z}{z_j}) - p_n \left( \frac{z}{z_j} \right) \right| \leq \frac{1}{n+1} \cdot \frac{|z|^{n+1}}{|z_j|} + \frac{1}{n+2} \cdot \frac{|z|^{n+2}}{|z_j|^2} + \ldots \leq \frac{1}{n+1} \cdot \frac{|z/z_j|^{n+1}}{1 - |z/z_j|} \leq 2 \cdot \frac{|z/z_j|^{n+1}}{n+1}
\]

Thus, we want a sequence of positive integers \( n_j \) such that

\[
\sum_{|z_j| \geq 2R} \frac{|z/z_j|^{n_j+1}}{n_j+1} < \infty \quad \text{(with } |z| < R)\]

Of course, the choice of \( n_j \)'s must be compatible with enlarging \( R \), but this is easily arranged. For example, \( n_j = j - 1 \) succeeds:

\[
\sum_j \left| \frac{z}{z_j} \right|^j = \sum_{|z_j| < 2R} \left| \frac{z}{z_j} \right|^j + \sum_{|z_j| \geq 2R} \left| \frac{z}{z_j} \right|^j \leq \sum_{|z_j| < 2R} \left| \frac{z}{z_j} \right|^j + \sum_j 2^{-j}
\]

Since \( \{z_j\} \) is discrete, the sum over \( |z_j| < 2R \) is finite, so we have convergence, and convergence of the infinite product with \( n_j = j \):

\[
\prod_j \left( 1 - \frac{z}{z_j} \right) e^{p_j(z/z_j)} = \prod_j \left( 1 - \frac{z}{z_j} \right) \exp \left( \frac{z}{z_j} + \frac{z^2}{2z_j} + \frac{z^3}{3z_j} + \ldots + \frac{z^j}{jz^j} \right)
\]
[1.2] Canonical products and genus

Given entire \( f \) with zeros \( z_j \neq 0 \) and a zero of order \( m \) at 0, ratios

\[
\varphi(z) = \frac{f(z)}{z^m \prod_j (1 - \frac{z}{z_j}) \cdot e^{p_n(z/z_j)}}
\]

with convergent infinite products are entire, and do not vanish. Non-vanishing entire \( \varphi \) has an entire logarithm:

\[
g(z) = \log \varphi(z) = \int_0^z \frac{\varphi'(\zeta)}{\varphi(\zeta)} d\zeta
\]

Thus, non-vanishing entire \( \varphi \) is expressible as

\[
\varphi(z) = e^{g(z)} \quad \text{(with } g \text{ entire)}
\]

Thus, the most general entire function with prescribed zeros is of the form

\[
f(z) = e^{g(z)} \cdot z^m \prod_j (1 - \frac{z}{z_j}) \cdot e^{p_n(z/z_j)} \quad \text{(with } g \text{ entire)}
\]

Naturally, with fixed \( f \), altering the \( n_j \) necessitates a corresponding alteration in \( g \).

We are most interested in zeros \( \{ z_j \} \) allowing a uniform integer \( h \) giving convergence of the infinite product in an expression

\[
f(z) = e^{g(z)} \cdot z^m \prod_j \left(1 - \frac{z}{z_j}\right) e^{p_h(z/z_j)} = z^m \prod_j \left(1 - \frac{z}{z_j}\right) \exp \left(\frac{z}{z_j} + \frac{z^2}{2z_j^2} + \frac{z^3}{3z_j^3} + \ldots + \frac{z^h}{h z^h}\right)
\]

When \( f \) admits a product expression with a uniform \( h \), a product expression with minimal uniform \( h \) is a canonical product for \( f \).

When, further, the leading factor \( e^{g(z)} \) for \( f \) has \( g(z) \) a polynomial, the genus of \( f \) is the maximum of \( h \) and the degree of \( g \).

2. Poisson-Jensen formula

Jensen’s formula and the Poisson-Jensen formula are essential in the difficult half of the Hadamard theorem (below) comparing genus of an entire function to its order of growth.

The logarithm \( u(z) = \log |f(z)| \) of the absolute value \( |f(z)| \) of a non-vanishing holomorphic function \( f \) on a neighborhood of the unit disk is harmonic, that is, is annihilated by \( \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \): expand

\[
\Delta \log |f(z)| = \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \left( \log f(z) + \frac{1}{2} \log F(z) \right)
\]

Conveniently, the two-dimensional Laplacian is the product of the Cauchy-Riemann operator and its conjugate. Since \( \log f \) is holomorphic and \( \log F \) is anti-holomorphic, both are annihilated by the product of the two linear operators. This verifies that \( \log |f(z)| \) is harmonic.

Thus, \( \log |f(z)| \) satisfies the mean-value property for harmonic functions:

\[
\log |f(0)| = u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta
\]
Next, let $f$ have zeros $\rho_j$ in $|z| < 1$ but none on the unit circle. We manufacture a holomorphic function $F$ from $f$ but without zeros in $|z| < 1$, and with $|F| = |f|$ on $|z| = 1$, by the standard ruse

$$F(z) = f(z) \cdot \prod_j \frac{1 - \overline{\rho}_j z}{z - \rho_j}$$

Indeed, for $|z| = 1$, the numerator of each factor has the same absolute value as the denominator:

$$|z - \rho_j| = \frac{1}{|z|} |1 - \overline{\rho}_j z| = |1 - \overline{\rho}_j z|$$

For simplicity, suppose no $\rho_j$ is 0. Applying the mean-value identity to $\log |F(z)|$ gives

$$\log |f(0)| - \sum_j \log |\rho_j| = \log |F(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |F(e^{i\theta})| \, d\theta = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta$$

and then the basic Jensen’s formula

$$\log |f(0)| - \sum_j \log |\rho_j| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \, d\theta \quad \text{(for } |\rho_j| < 1\text{)}$$

The Poisson-Jensen formula is obtained by replacing 0 by an arbitrary point $z$ inside the unit disk. This is obtained by replacing $f$ by $f \circ \varphi_z$, where $\varphi_z$ is a linear fractional transformation mapping 0 → $z$ and stabilizing[1] the unit disk:

$$\varphi_z = \left( \begin{array}{cc} 1 & z \\ z & 1 \end{array} \right) : w \mapsto \frac{w + z}{\overline{z}w + 1}$$

This replaces the zeros $\rho_j$ by $\varphi_z^{-1}(\rho_j) = \frac{\rho_j - z}{\overline{z}\rho_j + 1}$. Instead of the mean-value property expressing $f(0)$ as an integral over the circle, use the Poisson formula (see appendix) for $f(z)$. This gives the basic Poisson-Jensen formula

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j - z}{-\overline{\rho}_j + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < 1, |\rho_j| < 1\text{)}$$

More generally, for holomorphic $f$ on a neighborhood of a disk of radius $r > 0$ with zeros $\rho_j/r$ in that disk, apply the previous to $f(r \cdot z)$ with zeros $\rho_j/r$ in the unit disk:

$$\log |f(r \cdot z)| - \sum_j \log \left| \frac{\rho_j/r - z}{-\overline{\rho}_j + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(r \cdot e^{i\theta})| \cdot \frac{1 - |z/r|^2}{|z - e^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < 1\text{)}$$

Replacing $z$ by $z/r$ gives

$$\log |f(z)| - \sum_j \log \left| \frac{\rho_j/r - z/r}{-\overline{\rho}_j + 1} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(\overline{r}e^{i\theta})| \cdot \frac{1 - |z/r|^2}{|z/r - e^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < r\text{)}$$

[1] To verify that such maps stabilize the unit disk, expand the natural expression:

$$1 - \left| \frac{w + z}{\overline{z}w + 1} \right|^2 = |\overline{w} + 1|^2 \cdot \left( |\overline{w} + 1|^2 - |w + z|^2 \right) = |\overline{w} + 1|^2 \cdot \left( |zw|^2 + \overline{z}w + \overline{w} + 1 - |w|^2 - \overline{z}w - z^2 \right)$$

$$= |\overline{w} + 1|^2 \cdot \left( |zw|^2 + 1 - |w|^2 - |z|^2 \right) = |\overline{w} + 1|^2 \cdot (1 - |w|^2) \cdot (1 - |w|^2) > 0$$
which rearranges slightly to the general Poisson-Jensen formula

\[
\log |f(z)| - \sum_j \log \left| \frac{\rho_j - z}{z - \rho_j} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < r, |\rho_j| < r)\]

The case \(z = 0\) is the general Jensen formula for arbitrary radius \(r\):

\[
\log |f(0)| - \sum_j \log \left| \frac{\rho_j}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \quad \text{(with } |\rho_j| < r)\]

### 3. Hadamard products

The order of an entire function \(f\) is the smallest positive real \(\lambda\), if it exists, such that, for every \(\varepsilon > 0\),

\[
|f(z)| \leq e^{\lambda |z|^\varepsilon} \quad \text{(for all sufficiently large } |z|)\]

The connection to infinite products is:

**3.0.1 Theorem: (Hadamard)** The genus \(h\) and order \(\lambda\) are related by \(h \leq \lambda < h + 1\). In particular, one is finite if and only the other is finite.

**Proof:** First, the easier half. For \(f\) of finite genus \(h\) expressed as

\[
f(z) = e^{g(z)} \cdot z^m \prod_j \left(1 - \frac{z}{\rho_j} \right) e^{\rho_j h z/j} = e^{g(z)} \cdot z^m \prod_j \left(1 - \frac{z}{\rho_j} \right) \exp \left( \frac{z^2}{2\rho_j^2} + \frac{z^3}{3\rho_j^3} + \ldots + \frac{z^h}{h\rho_j^h} \right)
\]

the leading exponential has polynomial \(g\) of degree at most \(h\), so \(e^{g(z)}\) is of order at most \(h\). The order of a product is at most the maximum of the orders of the factors, so it suffices to prove that the order of the infinite product is at most \(h + 1\).

The assumption that \(h\) is the genus of \(f\) is equivalent to

\[
\sum_j \frac{1}{|\rho_j|^{h+1}} < \infty
\]

We use this to directly estimate the infinite product, showing that it has order of growth \(\lambda\) at most \(h + 1\).

We need an estimate on \(F_h(w) = (1 - w) e^{\rho_h w}\) applicable for all \(w\), not merely for \(|w| < 1\). We collect some inequalities. There is the basic

\[
\log |F_h(w)| = \log |(1 - w) e^{\rho_{h-1} w} e^{w^h / h}| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \quad \text{(for all } w)\]

As before, for \(|w| < 1\),

\[
\log |F_h(w)| \leq \frac{1}{h + 1} \cdot |w|^{h+1} + \frac{1}{h + 2} \cdot |w|^{h+2} + \ldots \leq |w|^{h+1} \cdot \frac{1}{1 - |w|} \quad \text{(for } |w| < 1)\]

This gives \((1 - |w|) \cdot \log |F_h(w)| \leq |w|^{h+1}\) for \(|w| < 1\). Adding to the latter the basic relation multiplied by \(|w|\) gives

\[
\log |F_h(w)| \leq |w| \cdot \log |F_{h-1}(w)| + (1 + \frac{1}{h}) |w|^{h+1} \quad \text{(for } |w| < 1)\]
In fact, the latter inequality also holds for $|w| \geq 1$ and $\log |F_{h-1}(w)| \geq 0$, from the basic relation. For $\log |F_{h-1}(w)| < 0$ and $|w| \geq 1$, from the basic relation,

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \leq \frac{|w|^h}{h} \leq \left(1 + \frac{1}{h}\right)|w|^{h+1} \quad \text{(for $\log |F_{h-1}(w)| < 0$ and $|w| \geq 1$)}$$

Now prove $\log |F_h(w)| \ll_h |w|^{h+1}$, by induction on $h$. For $h = 0$, from $\log |x| \leq |x| - 1$,

$$\log |1 - w| \leq |1 - w| - 1 \leq |w| - 1 = |w|$$

Assume $\log |F_{h-1}(w)| \ll_h |w|^h$. For $|w| < 1$, we reach the desired conclusion by

$$\log |F_h(w)| \leq |w| \cdot \log |F_{h-1}(w)| + \left(1 + \frac{1}{h}\right)|w|^{h+1} \ll_h |w| \cdot |w|^h + \left(1 + \frac{1}{h}\right)|w|^{h+1} \quad \text{(for $|w| < 1$)}$$

For $|w| \geq 1$ and $\log |F_{h-1}(w)| > 0$, from the basic relation

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \ll_h |w|^h + \frac{|w|^h}{h} \ll_h |w|^{h+1} \quad \text{(for $|w| \geq 1$ and $\log |F_{h-1}(w)| > 0$)}$$

For $\log |F_{h-1}(w)| \leq 0$ and $|w| \geq 1$, from the basic relation we already have

$$\log |F_h(w)| \leq \log |F_{h-1}(w)| + \frac{|w|^h}{h} \ll_h |w|^h + \frac{|w|^h}{h} \ll_h |w|^{h+1} \quad \text{(for $|w| \geq 1$ and $\log |F_{h-1}(w)| < 0$)}$$

This proves $\log |F_h(w)| \ll_h |w|^{h+1}$ for all $w$.

Estimate the infinite product:

$$\log \left| \prod_j (1 - \frac{z}{z_j}) \cdot e^{p_h(z_j)} \right| = \sum_j \log \left| (1 - \frac{z}{z_j}) \cdot e^{p_h(z_j)} \right| \ll_h \sum_j \left| \frac{z}{z_j} \right|^{h+1} < \infty$$

since $\sum 1/|z_j|^{h+1}$ converges. Thus, such an infinite product has growth order $\lambda \leq h + 1$.

Now the difficult half of the proof. Let $h \leq \lambda < h + 1$. Jensen’s formula will show that the zeros $z_j$ are sufficiently spread out for convergence of $\sum 1/|z_j|^{h+1}$. Without loss of generality, suppose $f(0) \neq 0$. From

$$\log |f(0)| - \sum_j \log \left| \frac{z_j}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \, d\theta \quad \text{(with $|z_j| < r$)}$$

certainly

$$\sum_{|z_j| < r/2} \log 2 \leq \sum_{|z_j| < r/2} -\log \left| \frac{z_j}{r} \right| \ll \varepsilon - \log |f(0)| + \frac{1}{2\pi} \int_0^{2\pi} r^{\lambda+\varepsilon} \, d\theta \ll r^{\lambda+\varepsilon} \quad \text{(for every $\varepsilon > 0$)}$$

With $\nu(r)$ the number of zeros inside the disk of radius $r$, this gives

$$\lim_{r \to +\infty} \frac{\nu(r)}{r^{\lambda+\varepsilon}} = 0 \quad \text{(for all $\varepsilon > 0$)}$$

Order the zeros by absolute value: $|z_1| \leq |z_2| \leq \ldots$ and for simplicity suppose no two have the same size. Then $j = \nu(|z_j|) \ll \varepsilon |z|^{\lambda+\varepsilon}$. Thus,

$$\sum \frac{1}{|z_j|^{h+1}} \ll \varepsilon \sum \frac{1}{(j^{1/\pi})^{h+1}} = \sum \frac{1}{j^{h+1/\pi}}$$
Thus, for $\frac{h+1}{\lambda + \varepsilon} > 1$, that is, for $\lambda + \varepsilon < h + 1$. When $\lambda < h + 1$, there is $\varepsilon > 0$ making such an equality hold.

It remains to show that the entire function $g(z)$ in the leading exponential factor is of degree at most $h + 1$, by showing that its $(h + 1)^{th}$ derivative is 0.

In the Poisson-Jensen formula

$$
\log |f(z)| - \sum_{|z_j| < r} \log \left| \frac{z - z_j}{-\bar{z}_j/r + r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < r)$$

application of $\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}$ annihilates the anti-holomorphic parts, returning us to an equality of holomorphic functions, as follows. The effect on the integrand is

$$
2 \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \frac{r^2 - |z|^2}{|z - re^{i\theta}|^2} = \frac{2}{(z - re^{i\theta})(\bar{z} - re^{-i\theta})} - \frac{r^2 - |z|^2}{(z - re^{i\theta})^2(\bar{z} - re^{-i\theta})} = 2 - \frac{|z|^2 + \bar{z}re^{i\theta} - r^2 + |z|^2}{(z - re^{i\theta})^2(\bar{z} - re^{-i\theta})} = 2 \frac{re^{i\theta}}{(z - re^{i\theta})^2}
$$

Thus,

$$
\frac{f'(z)}{f(z)} - \sum_{|z_j| < r} \frac{1}{z - z_j} + \sum_{|z_j| < r} \frac{\bar{z}_j}{\bar{z}_j/z - r^2} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^2} \, d\theta
$$

Further differentiation $h$ times in $z$ gives

$$
\left( \frac{f'(z)}{f(z)} \right)^{(h)} = \sum_{|z_j| < r} \frac{(-1)^h h!}{(z - z_j)^{h+1}} - \sum_{|z_j| < r} \frac{(-1)^h h! \cdot \bar{z}_j^{h+1}}{(\bar{z}_j z - r^2)^{h+1}} + \frac{(-1)^h (h + 1)!}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^{h+2}} \, d\theta
$$

We claim that the second sum and the integral go to 0 as $r \to +\infty$.

Regarding the integral, Cauchy’s integral formula gives

$$
\int_0^{2\pi} \frac{re^{i\theta}}{(z - re^{i\theta})^{h+2}} \, d\theta = 0
$$

Thus, letting $M_r$ be the maximum of $|f|$ on the circle of radius $r$, and taking $|z| < r/2$, up to sign the integral is

$$
\int_0^{2\pi} \log \left( \frac{M_r}{|f(re^{i\theta})|} \right) \cdot \frac{2re^{i\theta}}{(z - re^{i\theta})^{h+2}} \, d\theta \ll \frac{1}{r^{h+1}} \int_0^{2\pi} \log \left( \frac{M_r}{|f(re^{i\theta})|} \right) \, d\theta \ll \frac{r^{\lambda + \varepsilon}}{r^{h+1}} \cdot \int_0^{2\pi} -\log |f(re^{i\theta})| \, d\theta
$$

Jensen’s formula gives

$$
\frac{1}{2\pi} \int_0^{2\pi} -\log |f(re^{i\theta})| \, d\theta \leq -\log |f(0)|
$$

Thus, for $\lambda + \varepsilon < h + 1$ the integral goes to 0 as $r \to +\infty$.

For the second sum, again take $|z| < r/2$, so for $|z_j| < r$

$$
\left| \frac{\bar{z}_j^{h+1}}{(\bar{z}_j z - r^2)^{h+1}} \right| \leq \frac{|z_j|^{h+1}}{(r^2 - |z_j|^2)^{h+1}} \ll \frac{|z_j|^{h+1}}{r^{h+1}(r - |z_j|)^{h+1}} \ll \frac{1}{r^{h+1}}
$$

We already showed that the number $\nu(r)$ of $|z_j| < r$ satisfies $\lim \nu(r)/r^{h+1} = 0$. Thus, this sum goes to 0 as $r \to +\infty$. Thus, taking the limit,

$$
\left( \frac{f'}{f} \right)^{(h)} = (-1)^h h! \sum_j \frac{1}{(z - z_j)^{h+1}}
$$
Returning to \( f(z) = e^{g(z)} \prod_j (1 - \frac{z}{z_j}) \cdot e^{p_j(z/z_j)} \), taking logarithmic derivative gives

\[
\frac{f'}{f} = g' + \sum_j \left( \frac{1}{z - z_j} + \frac{p'_j(z/z_j)}{z_j} \right)
\]

and taking \( h \) further derivatives gives

\[
\left( \frac{f'}{f} \right)^{(h)} = g^{(h+1)} + \sum_j \frac{(-1)^h h!}{(z - z_j)^{h+1}}
\]

Since the \( h^{th} \) derivative of \( f'/f \) is the latter sum, \( g^{(h+1)} = 0 \), so \( g \) is a polynomial of degree at most \( h \).

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### 4. Appendix: harmonic functions

We recall the mean-value property and Poisson’s formula for harmonic functions. The Laplacian is

\[
\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

and continuously twice-differentiable functions \( u \) with \( \Delta u = 0 \) are harmonic.

**[4.0.1] Theorem:** (Mean-value property) For harmonic \( u \) on a neighborhood of the unit disk,

\[
u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta} \cdot z) \, d\theta
\]

**Proof:** Consider the rotation-averaged function

\[
v(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta} \cdot z) \, d\theta \quad \text{(for } |z| \leq 1\text{)}
\]

Since the Laplacian \( \Delta \) is rotation-invariant, \( v \) is a rotation-invariant harmonic function. In polar coordinates, for rotation-invariant functions \( v(z) = f(|z|) \), the Laplacian is

\[
\Delta v = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f(\sqrt{x^2 + y^2}) = \frac{\partial}{\partial x} \left( \frac{x}{|z|} f'(|z|) \right) + \frac{\partial}{\partial y} \left( \frac{y}{|z|} f'(|z|) \right)
\]

\[
= \frac{1}{|z|} f' - \frac{x^2}{|z|^3} f'' + \frac{x^2}{|z|^2} f' + \frac{1}{|z|} f' - \frac{y^2}{|z|^3} f' + \frac{y^2}{|z|^2} f'' = f'' + \frac{1}{|z|} f'
\]

The ordinary differential equation \( f'' + f'/r = 0 \) on an interval \((0, R)\) is an equation of Euler type, meaning expressible in the form \( v^2 f'' + B r f' + C f = 0 \) with constants \( B, C \). In general, such equations are solved by letting \( f(r) = r^\lambda \), substituting, dividing through by \( r^\lambda \), and solving the resulting indicial equation for \( \lambda \):

\[
\lambda(\lambda - 1) + A\lambda + B = 0
\]

Distinct roots \( \lambda_1, \lambda_2 \) of the indicial equation produce linearly independent solutions \( r^{\lambda_1} \) and \( r^{\lambda_2} \). However, as in the case at hand, a repeated root \( \lambda \) produces a second solution \( r^\lambda \cdot \log r \).

Here, the indicial equation is \( \lambda^2 = 0 \), so the general solution is \( a + b \log r \).
When \( b \neq 0 \), the solution \( a + b \log r \) blows up as \( r \to 0^+ \). Since \( f(0) = v(0) = u(0) \) is finite, it must be that \( b = 0 \). Thus, a rotation-invariant harmonic function on the disk is constant. Thus, its average over a circle is its central value. This proves the mean-value theorem for harmonic functions.

\[ 4.0.2 \text{ Remark:} \] The solutions \( a + b \log r \) do indeed exhaust the possible solutions: given \( f'' + f'/r = 0 \) on \((0, R)\),

\[
\frac{\partial}{\partial r} (r \cdot f') = r \cdot f'' + f' = r \cdot (-f'/r) + f' = 0
\]

Thus, \( r \cdot f' \) is constant, and so on.

With some computation, from mean-value property we will obtain

\[ 4.0.3 \text{ Theorem:} \text{(Poisson's formula)} \] For \( u \) harmonic on a neighborhood of the unit disk,

\[
u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \cdot \frac{1 - |z|^2}{|z - e^{i\theta}|^2} \, d\theta \quad \text{(for } |z| < 1)\]

\[ \text{Proof:} \] Composition with holomorphic maps preserves harmonic-ness. With \( \varphi_z \) the linear fractional transformation given by matrix \( \varphi_z \sim \begin{pmatrix} 1 & z \\ z & 1 \end{pmatrix} \), the mean-value property for \( u \circ \varphi_z \) gives

\[
u(z) = (u \circ \varphi_z)(0) = \frac{1}{2\pi} \int_0^{2\pi} (u \circ \varphi_z)(e^{i\theta}) \, d\theta
\]

Linear fractional transformations stabilizing the unit disk map the unit circle to itself. Replace \( e^{i\theta} \) by \( e^{i\theta'} = \varphi_z^{-1}(e^{i\theta}) \)

\[
u(z) = (u \circ \varphi_z)(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) \, d\theta'
\]

Computing the change of measure will yield the Poisson formula. This is computed by

\[
ie^{i\theta'} \cdot \frac{\partial e^{i\theta'}}{\partial \theta} = \frac{\partial}{\partial \theta} e^{i\theta'} = \frac{ie^{i\theta}}{1 - \overline{z}e^{i\theta}} + \frac{i\overline{z}e^{i\theta}(e^{i\theta} - z)}{(1 - \overline{z}e^{i\theta})^2} = \frac{ie^{i\theta} - i\overline{z}e^{2i\theta} + i\overline{z}e^{2i\theta} - ie^{i\theta}|z|^2}{(1 - \overline{z}e^{i\theta})^2} = \frac{ie^{i\theta}(1 - |z|^2)}{(1 - \overline{z}e^{i\theta})^2}
\]

Thus,

\[
\frac{\partial e^{i\theta'}}{\partial \theta} = \frac{1}{e^{i\theta'}(1 - \overline{z}e^{i\theta})^2} \cdot \frac{ie^{i\theta}(1 - |z|^2)}{e^{i\theta}(1 - \overline{z}e^{i\theta})} = \frac{1 - |z|^2}{|z - e^{i\theta}|^2}
\]

giving the asserted integral.


