

(August 29, 2013)

Counting zeros of $\zeta(s)$

Paul Garrett garrett@math.umn.edu <http://www.math.umn.edu/~garrett/>

[This document is http://www.math.umn.edu/~garrett/m/mfms/notes_2013-14/counting_zeros_of_zeta.pdf]

Zeros of $\zeta(s)$ in the critical strip $0 \leq \operatorname{Re}(s) \leq 1$ are counted [1] using the *argument principle* and the Laplace-Stirling asymptotic

$$\Gamma(s) = (s - \frac{1}{2}) \log s - s + O(1) \quad (\text{in } \operatorname{Re}(s) \geq \delta > 0 \text{ as } |s| \rightarrow \infty)$$

The counting function of interest is [2]

$$N(T) = \text{number of zeros of } \zeta(s) \text{ in } 0 \leq \operatorname{Im}(s) \leq T \text{ and } 0 \leq \operatorname{Re}(s) \leq 1$$

By the *argument principle* [3]

$$N(T) = \frac{1}{2} \cdot \left(\frac{1}{2\pi i} \int_{R_T} \frac{\xi'(s)}{\xi(s)} ds \right) + O(1)$$

where R_T is the rectangle connecting $2 \pm iT$ and $-1 \pm iT$, traversed counter-clockwise, deformed slightly to skirting any zeros of $\zeta(s)$. The division by 2 takes into account the double-counting of zeros off the real interval $[0, 1]$, and the $O(1)$ term accommodates miscounting poles at $s = 0, 1$ and any zeros on $[0, 1]$. The proof of the following is simply an estimate of this integral, specifically, giving the leading terms in an asymptotic in T .

[0.0.1] Theorem:

$$N(T) = \frac{1}{2\pi} \cdot T \log T - \frac{\log 2\pi e}{2\pi} \cdot T + O(\log T) = \frac{1}{2\pi} T \log \frac{T}{2\pi e} + O(\log T)$$

[0.0.2] Remark: In particular, there are infinitely-many zeros of $\zeta(s)$ in the critical strip.

[0.0.3] Remark: The vertical asymptotics of $\Gamma(s)$ dominate and completely determine the leading terms of the asymptotic expansion, by a direct computation which determines the relevant constants.

Proof: Using the functional equation $\xi(1-s) = \xi(s)$, and the symmetry $\xi(\bar{s}) = \overline{\xi(s)}$, we integrate only upward from 2 to $2 + iT$, and then left from $2 + iT$ to $\frac{1}{2} + iT$.

The argument-principle integral computes $1/2\pi$ times the net change in the imaginary part of $\log \xi(s)$ over the given paths, requiring *continuity* of the logarithm. We compute separately the net changes of the imaginary parts of the summands in

$$\log \xi(s) = -\frac{s}{2} \log \pi + \log \Gamma\left(\frac{s}{2}\right) + \log \zeta(s)$$

[1] From [Backlund 1914, 1916, 1918]. See also [Titchmarsh/Heath-Brown 1951/1989] pages 212-213. [Backlund 1916] was a thesis done under E. Lindelöf's supervision.

[2] The convergent Euler product shows that there are no zeros in $\operatorname{Re}(s) > 1$. The functional equation $\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$ shows that the only zeros of $\zeta(s)$ in $\operatorname{Re}(s) < 0$ are where $\Gamma\left(\frac{s}{2}\right)$ has poles, namely, negative even integers. These are the *trivial* zeros of $\zeta(s)$.

[3] As usual, $\xi(s)$ is the completed zeta function $\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$. The usual notation is $S(T) = \frac{1}{\pi} \arg \zeta(s)$, required to be 0 at $s = 2$, and continuous along the vertical line from 2 to $2 + iT$ and then to $\frac{1}{2} + iT$. When there is a zero along $(0, 1) + iT$, compute $S(T)$ slightly above T .

Obviously the net change of imaginary part of the logarithm of $\pi^{-s/2}$ is

$$\operatorname{Im}(\log \pi^{-(\frac{1}{2}+iT)/2} - \log \pi^{-2/2}) = \operatorname{Im}\left(-\frac{\frac{1}{2}+iT}{2} \log \pi\right) = -\frac{T}{2} \log \pi$$

From

$$\log \Gamma(s) = (s - \frac{1}{2}) \log s - s + O(1)$$

we have

$$\log \Gamma(\frac{s}{2}) = (\frac{s}{2} - \frac{1}{2}) \log \frac{s}{2} - \frac{s}{2} + O(1)$$

Thus, the net change from 2 to $\frac{1}{2} + iT$ is

$$\begin{aligned} \operatorname{Im}\left(\log \Gamma\left(\frac{\frac{1}{2}+iT}{2}\right) - \log \Gamma\left(\frac{2}{2}\right)\right) &= \operatorname{Im}\left(\left(\frac{\frac{1}{2}+iT}{2} - \frac{1}{2}\right) \log \frac{\frac{1}{2}+iT}{2} - \frac{\frac{1}{2}+iT}{2}\right) + O(1) \\ &= \operatorname{Im}\left(\left(-\frac{1}{4} + \frac{iT}{2}\right)\left(\frac{\pi i}{2} + \log \frac{T}{2} + O\left(\frac{1}{T}\right)\right)\right) - \frac{T}{2} + O(1) = \frac{T}{2} \log \frac{T}{2} - \frac{T}{2} + O(1) \end{aligned}$$

Since $s = 2 + i\mathbb{R}$ is nicely within the region of absolute convergence of the Euler product, $\log \zeta(2 + it)$ is *bounded* on that line, so the net change in the imaginary part of the argument of $\zeta(s)$ from 2 to $2 + iT$ is $O(1)$.

The subtle computation concerns the net change in the argument of $\zeta(s)$ from $2 + iT$ to $\frac{1}{2} + iT$. We recall a version of part of a relevant lemma from [Titchmarsh/Heath-Brown 1951/1989], page 213, which uses *Jensen's Lemma* to approximate the number of zeros, hence, the change in argument, in terms of the *growth*, of a function:

[0.0.4] Lemma: Let f be a holomorphic function on a vertical strip $-1 \leq \sigma \leq 4$, except possibly for a simple pole at $s = 1$. Suppose that $f(\bar{s}) = \overline{f(s)}$. Assume a lower bound $|f(2 + it)| \geq m > 0$, and a family of upper bounds

$$|f(\sigma + it)| \leq M(T) \quad (\text{for } \frac{1}{4} \leq \sigma \leq 4 \text{ and } 1 \leq t \leq T)$$

Then, for T not the vertical coordinate of a zero of f , there is the upper bound for change in argument from $2 + iT$ to $\frac{1}{2} + iT$

$$|\arg f(\frac{1}{2} + iT) - \arg f(2 + iT)| \leq \frac{\pi}{\log((2 - \frac{1}{4})/(2 - \frac{1}{2}))} \cdot \left(\log M(T + 2) + \log \frac{1}{m}\right) + \pi$$

[0.0.5] Remark: Naturally, some of the details are insignificant, being mere artifacts of the proof. At the same time, we give a more specific version of the result than [Titchmarsh/Heath-Brown 1951/1989].

Proof: Let q be the number of vanishings of $\operatorname{Re} f(\sigma + iT)$ between $2 + iT$ and $\frac{1}{2} + iT$. The vanishings divide the interval into $q + 1$ subintervals on each of which either $\operatorname{Re} f \geq 0$ or $\operatorname{Re} f \leq 0$. In particular, the value of f stays in either the right or left half-plane, so the $\arg f$ cannot change more than π in each subinterval. Thus,

$$|\arg f(\frac{1}{2} + iT) - \arg f(2 + iT)| \leq (q + 1) \cdot \pi$$

Using $f(\bar{s}) = \overline{f(s)}$, the count q is the number of zeros of $g(z) = \frac{1}{2}[f(z + iT) + f(z - iT)]$ on the real interval $\frac{1}{2} \leq z \leq 2$. Certainly this count is bounded by the number of zeros of $g(z)$ in the disk $|z - 2| \leq 2 - \frac{1}{2}$.

Let $\nu(r)$ be the number of zeros in $|z - 2| \leq r$. Setting up application of Jensen's lemma,^[4] we have an upper bound for q :

$$\int_0^{2-\frac{1}{4}} \frac{\nu(r)}{r} dr \geq \int_{2-\frac{1}{2}}^{2-\frac{1}{4}} \frac{\nu(r)}{r} dr \geq \nu(2 - \frac{1}{2}) \cdot \frac{2 - \frac{1}{4}}{2 - \frac{1}{2}} \geq q \cdot \frac{2 - \frac{1}{4}}{2 - \frac{1}{2}}$$

Jensen's lemma leads to an upper bound for the integral:

$$\int_0^{2-\frac{1}{4}} \frac{\nu(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log |g(2 + (2 - \frac{1}{4})e^{i\theta})| d\theta - \log |g(2)| \leq \log M(T + 2) + \log \frac{1}{m}$$

giving the lemma. ///

The lemma applies to $f(s) = \zeta(s)$ since the convergent Euler product is bounded away from 0 on $2 + i\mathbb{R}$, with bound $M(t) = t^N$ on a given vertical strip. Thus, the net change in the argument of $\zeta(s)$ from $2 + iT$ to $\frac{1}{2} + iT$ is $O(\log T)$.

Altogether, the argument principle gives

$$\begin{aligned} N(T) &= \frac{1}{2} \cdot \frac{1}{2\pi} \cdot 4 \cdot \left(\frac{T}{2} \log \frac{T}{2} - \frac{T}{2} - \frac{T}{2} \log \pi \right) + O(\log T) \\ &= \frac{1}{\pi} \cdot \left(\frac{T}{2} \log \frac{T}{2} - \frac{T}{2} (1 + \log \pi) \right) + O(\log T) = \frac{T \log T}{2\pi} - \frac{T}{2\pi} \log 2 - \frac{T}{2\pi} (1 + \log \pi) + O(\log T) \\ &= \frac{1}{2\pi} \cdot T \log T - \frac{\log 2\pi e}{2\pi} \cdot T + O(\log T) \end{aligned}$$

which is the asserted asymptotic. ///

Bibliography:

[Backlund 1914] R.J. Backlund, *Sur les zéros de la fonction $\zeta(s)$ de Riemann*, C.R. **158** (1914), 1979-81.

[Backlund 1916] R.J. Backlund, *Über die Nullstellen der Riemannschen Zetafunktion*, Dissertation, Helsingfors, 1916.

[Backlund 1918] R.J. Backlund, *Über die Nullstellen der Riemannschen Zetafunktion*, Acta Math. **41** (1918), 345-75.

[Titchmarsh/Heath-Brown 1951/1989] E.C. Titchmarsh, *The theory of the Riemann zeta function*, Oxford University Press, 1951. Second edition, revised by D.R. Heath-Brown, 1986.

[4] *Jensen's Lemma* usually appears as follows: for holomorphic f on $|z| \leq r$, no zeros on $|z| = r$, and $f(0) \neq 0$,

$$\log |f(0)| - \sum_{\rho} \log \left| \frac{\rho}{r} \right| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \quad (\text{summed over zeros } |\rho| < r \text{ of } f)$$

Letting $\nu(t)$ be the number of zeros of size less than t ,

$$- \sum_{\rho} \log \left| \frac{\rho}{r} \right| = \sum_{\rho} (\log r - \log |\rho|) = \sum_{\rho} \int_{|\rho|}^r \frac{dt}{t} = \int_0^r \nu(t) \frac{dt}{t}$$