Phragmén-Lindelöf Theorems

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The maximum modulus principle [1] in complex analysis does not apply to unbounded regions. That is, holomorphic functions on an unbounded region may be bounded by 1 on the edges but be severely unbounded in the interior.

A simple example is $f(z) = e^{e^z}$: for $z$ real and going to $+\infty$ this function blows up. Indeed,

$$
|e^{e^{x+iy}}| = e^{\Re(e^{x+iy})} = e^{e^x \cdot \cos y}
$$

Thus, for fixed $y = \Im z$ with $\cos y > 0$, the function blows up as $x = \Re z \to +\infty$. On the other hand, for $\cos y = 0$ the function is bounded. Thus, on the strip $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, the function $e^{e^x}$ is bounded on the edges but blows up as $x \to +\infty$.

This example suggests *growth conditions* under which a bound of 1 on the edges implies the same bound throughout the strip.

[1.0.1] **Theorem:** Let $f$ be a holomorphic function on the horizontal half-strip

$$
\{z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \text{ and } 0 \leq x\}
$$

If

$$
|f(z)| \ll e^{C \Re z} \quad \text{(for some constant } 0 \leq C < 1)\]

then $|f(z)| \leq 1$ on the edges of the half-strip implies $|f(z)| \leq 1$ in the interior, as well.

**Proof:** Unsurprisingly, the proof is a reduction to the usual maximum modulus principle. Take any fixed $D$ in the range

$$
C < D < 1
$$

The function

$$
F_\varepsilon(z) = \frac{f(z)}{e^{\varepsilon D z}} \quad \text{(for } \varepsilon > 0)\]

is bounded by 1 on the edges of the half-strip, and in the interior goes to 0 uniformly in $y$ as $x \to +\infty$, for fixed $\varepsilon > 0$. This uniform decay in the interior uses the modification with $D$. Thus, on a rectangle

$$
R_T = \{z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \text{ and } 0 \leq x \leq T\}
$$

for sufficiently large $T > 0$ depending upon $\varepsilon$, the function $F_\varepsilon$ is bounded by 1 on the edge. The usual maximum modulus principle implies that $F_\varepsilon$ is bounded by 1 throughout. That is, for each fixed $z_o$ in the half-strip,

$$
|f(z_o)| \leq e^{\varepsilon \cdot e^{D \Re z_o}} \quad \text{(for all } \varepsilon > 0)\]

We can let $\varepsilon \to 0^+$, giving $|f(z_o)| \leq 1$. ///

[1] The *maximum modulus principle* in complex analysis is that a holomorphic function $f$ on a bounded region in $\mathbb{C}$ with $|f(z)| \leq 1$ on the edges is bounded by 1 in the interior, as well.
[1.0.2] **Remark:** Analogous theorems on strips of other widths follow by using $e^{c\Re z}$ with suitable constants $c$.

An analogous theorem on a full strip, rather than half-strip, follows by using a function like $e^{\cosh z}$ in place of $e^{\Re z}$, as follows.

[1.0.3] **Theorem:** Let $f$ be a holomorphic function on the full horizontal strip

$$\{z : -\frac{\pi}{2} \leq \Im z \leq \frac{\pi}{2}\}$$

If

$$|f(z)| \ll e^{\cosh C \Re z}$$

(for some constant $0 \leq C < 1$) then $|f(z)| \leq 1$ on the edges of the strip implies $|f(z)| \leq 1$ in the interior, as well.

**Proof:** Again, reduce to the maximum modulus principle. Fix $D$ in the range $C < D < 1$. The function

$$F_\varepsilon(z) = f(z)/e^{\varepsilon \cosh D z}$$

(for $\varepsilon > 0$) is bounded by 1 on the edges of the strip, and in the interior goes to 0 uniformly in $y$ as $x \to \pm \infty$, for fixed $\varepsilon > 0$. Thus, on a rectangle

$$R_T = \{z : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \text{ and } -T \leq x \leq T\}$$

(for large $T > 0$, depending upon $\varepsilon$)

the function $F_\varepsilon$ is bounded by 1 on the edge. The usual maximum modulus principle implies that $F_\varepsilon$ is bounded by 1 throughout. That is, for each fixed $z_0$ in the half-strip,

$$|f(z_0)| \leq e^{\varepsilon \cosh D \Re z_0}$$

(for all $\varepsilon > 0$)

We can let $\varepsilon \to 0^+$, giving $|f(z_0)| \leq 1$. 

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The details of various adjustments can be made to disappear by strengthening the hypotheses:

[1.0.4] **Corollary:** Let $f$ be a holomorphic function on a strip or half-strip, with a bound

$$|f(z)| \ll e^{|z|^A}$$

(for some $A > 0$)

If $|f(z)| \leq 1$ on the edges of the (half-)strip, then $|f(z)| \leq 1$ in the interior, as well.

[1.0.5] **Remark:** Further variations are easily possible, by additional adjustments of functions. For example, *polynomial growth* of a function $f$ on the edges of a strip or half-strip can be accommodated by considering $f(z)/(z - z_0)^M$ for $z_0$ outside the strip, and large $M$. //