Theorem: Every elliptic function (with lattice \( \Lambda \)) is expressible in terms of the corresponding \( \wp \) and \( \wp' \). That is, for lattice \( \Lambda \), the field of meromorphic \( \Lambda \)-periodic functions is exactly the collection of rational expressions in \( \wp_\Lambda(z) \) and \( \wp'_\Lambda(z) \). Further, all even \( \Lambda \)-periodic functions are rational expressions in \( \wp_\Lambda(z) \).

Incidental to the proof of the theorem, we have

Claim: Let \( f \) be a \( \Lambda \)-periodic meromorphic function. For a fixed choice of basis \( \omega_1, \omega_2 \) for \( \Lambda \), let \( F \) be the corresponding fundamental domain as above. Let \( z_1, \ldots, z_m \) be the zeros of \( f \) in \( F \), and let \( p_1, \ldots, p_n \) be the poles, both including multiplicities.\(^1\) Then \( m = n \). Further,

\[
\sum_i z_i - \sum_j p_j = 0 \mod \Lambda
\]

Proof: Integrating \( f/f' \) around the boundary of \( F \) (make minor adaptations in case a zero or pole happens to be exactly on that path) computes \( 2\pi i (m - n) \), by Cauchy’s residue theorem. On the other hand, by periodicity of \( f/f' \), and since we integrate on opposite edges of the parallelogram \( F \) in opposite directions, this integral is 0. Thus, \( m = n \).

Similarly, integrate \( z \cdot f'/f \) around the boundary of \( F \). On one hand, by Cauchy’s residue theorem this computes

\[
2\pi i \left( \sum_i z_i - \sum_j p_j \right)
\]

This time, since the function with the factor of \( z \) thrown in is not periodic, the integral is not 0. However, there is still some cancellation. The integral is

\[
-\omega_2 \int_0^{\omega_1} \frac{f'}{f} + \omega_1 \int_0^{\omega_2} \frac{f'}{f}
\]

One may easily overlook the fact that the two integrals are integer multiples of \( 2\pi i \), which follows from\(^2\)

\[
\int_0^{\omega_1} \frac{f'}{f} = \int_0^{\omega_1} \frac{d\log f}{d\zeta}
\]

and the fact that \( f(0) = f(\omega_i) \). That is, as \( \zeta \) goes from 0 to \( \omega_i \), the function \((\log f)(\zeta)\) traces out a closed path circling 0 some integer number of times, say \( k_i \). Then the integral is

\[
-\omega_2 \cdot 2\pi i k_1 + \omega_1 \cdot 2\pi i k_2 \in 2\pi i \cdot \Lambda
\]

 Cancelling the factor of \( 2\pi i \), equating the two outcomes gives

\[
\sum_i z_i - \sum_j p_j \in \Lambda
\]

\(^1\) Usually including multiplicities means that for a zero \( z_0 \) of order \( \ell \) the point \( z_0 \) is included \( \ell \) times on the list of zeros. That is, this list is a multiset, not an ordinary set, since ordinary sets (by their nature) do not directly keep track of multiple occurrences of the same element.

\(^2\) This is an instance of the Argument Principle.
Proof: Let $f$ be a $\Lambda$-periodic meromorphic function on $\mathbb{C}$. We can break $f$ into odd and even pieces by

$$f(z) = \frac{f(z) + f(-z)}{2} + \frac{f(z) - f(-z)}{2}$$

For $f$ odd, the function $\wp' \cdot f$ is even, so it suffices to prove that every even elliptic function is rational in $\wp$.

The previous claim has immediate implications for the values of $\wp$, which we use to form an expression in $\wp$ that will duplicate the zeros and poles of the given even $f$. Generally, for even $f$, since $f(-z) = f(z)$, for $2z_o \not\in \Lambda$ and $f(z_o) = 0$, then $f(-z_o) = 0$ and $z_o$ and $-z_o$ are distinct modulo $\Lambda$. For $2z_o \in \Lambda$, the oddness (and periodicity) of $f'$ yields

$$f'(z_o) = -f'(-z_o) = -f'(-z_o + 2z_o) = -f'(z_o)$$

so $f'(z_o) = 0$, and the order of the zero $z_o$ is at least 2.

In particular, by the previous claim, since $\wp(z) - \wp(a)$ has the obvious double pole on $\Lambda$, it has exactly two zeros, whose sum is 0 modulo $\Lambda$. Obviously $a$ itself is a 0, and for $a \not\in \frac{1}{2} \Lambda$ the unique (mod $\Lambda$) other zero is $-a$. And for $a \in \frac{1}{2} \Lambda$ it is a double zero of $\wp(z) - \wp(a)$.

Thus, for a zero $z_o \not\in \Lambda$ of $f$, the order of vanishing of $\wp(z - \wp(z_o))$ at all its zeros is at most that of $f$ at those zeros. Thus, by comparison to $f(z)$, the function

$$\frac{f(z)}{\wp(z) - \wp(z_o)}$$

has lost two zeros (either $z_o$ and $-z_o$ or a double zero at $z_o$). The double pole of $\wp(z) - \wp(z_o)$ at 0 makes $f(z)/(\wp(z) - \wp(z_o))$ have order of vanishing at 0 two more than that of $f(z)$. No new poles are introduced by such an alteration, nor any zeros off $\Lambda$. Thus, since there are only finitely-many zeros (modulo $\Lambda$), after finitely-many such modifications we have a function $g(z)$ with no zeros off $\Lambda$.

Next, we get rid of poles of $g(z)$ off $\Lambda$ by a similar procedure, repeatedly multiplying by factors $\wp(z) - \wp(z_i)$. Thus, for some list of points $z_i$ not in $\Lambda$, with positive and negative integer exponents $e_i$,

$$f(z) \cdot \prod_{i} (\wp(z) - \wp(z_i))^{e_i}$$

has no poles or zeros off $\Lambda$. From the previous discussion, this expression has no zeros or poles at all, and then is constant.

[0.0.3] Remark: There is at least one other way to construct doubly-periodic functions directions, due to Jacobi, who expressed doubly-periodic functions as ratios of entire functions (theta functions) which are genuinely singly-periodic with periods (for example) $\mathbb{Z}$, and nearly (but not quite) periodic in another direction. (Indeed, we saw just above that entire functions that are genuinely doubly-periodic are constant!)