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Special values, Laurent coefficients, algebraic relations

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1. Special values $\zeta(2n)$ in a Laurent expansion
2. Special values $L(2n, \chi)$ in Laurent expansion of $\wp(z)$

1. Special values $\zeta(2n)$ in a Laurent expansion

Via Liouville's theorem, by cancelling poles, and so on,

$$f(z) = \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2} \quad (\text{secretly } f(z) = \frac{\pi^2}{\sin^2 \pi z}, \text{ but we don't use this})$$

satisfies

$$f'^2 = 4f^2(f - \pi^2)$$

The Laurent coefficients of $f(z)$ at 0 have direct relations to the *special values* $\zeta(2), \zeta(4), \dots$, producing algebraic relations among these values, as follows.

Let $g(z) = f(z) - \frac{1}{z^2}$, so $g(z)$ is holomorphic at $z = 0$, and

$$\begin{aligned} g(z) &= g(0) + \frac{g'(0)}{1!}z + \frac{g''(0)}{2!}z^2 + \dots \\ &= \sum_{n \neq 0} \frac{1}{n^2} + \sum_{n \neq 0} \frac{-2}{1! \cdot n^3}z + \sum_{n \neq 0} \frac{(-2)(-3)}{2! \cdot n^4}z^2 + \sum_{n \neq 0} \frac{(-2)(-3)(-4)}{3! \cdot n^5}z^3 + \sum_{n \neq 0} \frac{(-2)(-3)(-4)(-5)}{4! \cdot n^6}z^4 + \dots \end{aligned}$$

In the odd-degree sums the $\pm n$ terms cancel, giving

$$f(z) = \frac{1}{z^2} + 2\zeta(2) + 6\zeta(4)z^2 + 10\zeta(6)z^4 + 14\zeta(8)z^6 + \dots$$

and

$$f'(z) = \frac{-2}{z^3} + 12\zeta(4)z + 40\zeta(6)z^3 + 84\zeta(8)z^5 + \dots$$

The simplified relation $f'^2 = 4f^2(f - \pi^2)$ from above gives a recursion to determine $\zeta(2n)$ from $\zeta(2), \zeta(4), \dots, \zeta(2n-2)$, for $2n \geq 6$, since all the Laurent coefficients of $0 = f'^2 - 4f^2(f - \pi^2)$ vanish: namely, the first/lowest-degree term involving $\zeta(2n)$ is the z^{2n-6} term

$$\begin{aligned} 0 &= 2 \cdot \frac{-2}{z^3} \cdot (4n-2)(2n-2)\zeta(2n) z^{2n-3} - 4 \cdot 3 \cdot \left(\frac{1}{z^2}\right)^2 \cdot (4n-2)\zeta(2n) z^{2n-2} + (\text{previous}) \\ &= -\left(4(2n-2) + 12\right)(4n-2) \cdot \zeta(2n) + (\text{previous}) = -(8n+4)(4n-2) \cdot \zeta(2n) + (\text{previous}) \end{aligned}$$

where *previous* is a polynomial involving $\zeta(2), \zeta(4), \dots, \zeta(2n-2)$.

In fact, given that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$, this approach can prove

[1.0.1] Claim: $\zeta(2n)/\pi^{2n}$ is *rational*, for $2n = 2, 4, 6, 8, \dots$

Proof: Rewrite the relation in terms of normalizations $\zeta(2m)/\pi^{2m}$. From the Laurent expansion,

$$\pi^{-2}f(z/\pi) = \frac{1}{z^2} + \frac{2\zeta(2)}{\pi^2} + \frac{6\zeta(4)}{\pi^4}z^2 + \frac{10\zeta(6)}{\pi^6}z^4 + \frac{14\zeta(8)}{\pi^8}z^6 + \dots$$

Replacing $f(z)$ by $F(z) = \pi^{-2}f(z/\pi)$ gives [1] $F'(z) = \pi^{-3}f'(z/\pi)$, and the relation $f'^2 = 4f^2(f - \pi^2)$ becomes

$$\pi^6 F'^2 = 4\pi^4 F^2(\pi^2 F - \pi^2)$$

giving a relation with *rational* coefficients:

$$F'^2 = 4F^2(F - 1)$$

This relation gives a recursion with *rational* coefficients for the values $\zeta(2n)/\pi^{2n}$. Non-vanishing of the coefficient of $\zeta(2n)$ at its first appearance was checked above, so the recursion does not collapse. ///

[1.0.2] **Remark:** The above discussion clumsily mirrors more direct expression of special values of ζ in terms of *Bernoulli numbers*, better seen via Riemann's keyhole/Hankel contour expression for $\zeta(-n)$ and the *functional equation* for $\zeta(s)$. Nevertheless, it has some interest as a warm-up for the following example.

2. Special values $L(2n, \chi)$ in Laurent expansion of $\wp(z)$

As the algebraic relation $f'^2 = 4f^2(f - \pi^2)$ for $f(z) = \sum_n 1/(z+n)^2$ gave relations among the Laurent coefficients of f involving special values $\zeta(2n)$, the Weierstraß relation $\wp'^2 = 4\wp^3 - 60g_2\wp - 140g_3$ gives relations among the Laurent coefficients of $\wp(z)$. These Laurent coefficients are less elementary than the special values $\zeta(2n)$. For special lattices these are special values of *Hecke L-functions*, discussed below.

With fixed lattice Λ ,

$$\wp(z) = \frac{1}{z^2} + \sum_{0 \neq \lambda \in \Lambda} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right) = \frac{1}{z^2} + (\text{holomorphic at } z=0)$$

With $g(z) = \wp(z) - \frac{1}{z^2}$,

$$\begin{aligned} g(z) &= g(0) + \frac{g'(0)}{1!}z + \frac{g''(0)}{2!}z^2 + \dots \\ &= \sum_{\lambda \neq 0} \left(\frac{1}{\lambda^2} - \frac{1}{\lambda^2} \right) + \sum_{\lambda \neq 0} \frac{-2}{1! \cdot \lambda^3} z + \sum_{\lambda \neq 0} \frac{(-2)(-3)}{2! \cdot \lambda^4} z^2 + \sum_{\lambda \neq 0} \frac{(-2)(-3)(-4)}{3! \cdot \lambda^5} z^3 + \sum_{\lambda \neq 0} \frac{(-2)(-3)(-4)(-5)}{4! \cdot \lambda^6} z^4 + \dots \end{aligned}$$

In the odd-degree sums the $\pm\lambda$ terms cancel, so

$$\wp(z) = \frac{1}{z^2} + \sum_{\lambda \neq 0} \frac{(-2)(-3)}{2! \cdot \lambda^4} z^2 + \sum_{\lambda \neq 0} \frac{(-2)(-3)(-4)(-5)}{4! \cdot \lambda^6} z^4 + \dots = \frac{1}{z^2} + \sum_{n \geq 2} (2n-1)g_n z^{2n-2}$$

with $g_n = \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^{2n}}$. The Weierstraß relation gives a recursion for g_4, g_5, \dots in terms of g_2, g_3 : the lowest-degree coefficient in which g_n appears is that of z^{2n-6} , and this is

$$\begin{aligned} 0 &= \left(2 \cdot \frac{-2}{z^3} \cdot (2n-2)(2n-1)g_n z^{2n-3} \right) - \left(3 \cdot \frac{1}{z^4} \cdot (2n-1)g_n z^{2n-2} \right) + (\text{previous}) \\ &= -(2n-1) \left(4(2n-2) + 3 \right) g_n z^{2n-6} + (\text{previous}) = -(2n-1)(8n-5)g_n z^{2n-6} + (\text{previous}) \end{aligned}$$

The coefficient of g_n is non-zero, so g_n is a polynomial in g_2, g_3, \dots, g_{n-1} with rational coefficients, independent of the lattice.

[1] Since $f(z) = \sum \frac{1}{(z+n)^2} = \frac{\pi^2}{\sin^2 \pi z}$, in fact $F(z) = \frac{1}{\sin^2 z}$.

[2.1] **Some Hecke L -functions** The lattice $\Lambda = \mathbb{Z} \cdot i + \mathbb{Z}$ is the ring $\mathbb{Z}[i]$ of Gaussian integers. It is *Euclidean*, so is a principal ideal domain. The Galois norm is $N(a + bi) = a^2 + b^2$, and *units* in $\mathbb{Z}[i]$ must have norm ± 1 , so the only units are $\pm 1, \pm i$.

The *Dedekind zeta function* for $\mathfrak{o} = \mathbb{Z}[i]$ is

$$\zeta_{\mathfrak{o}}(s) = \sum_{0 \neq \alpha \in \mathfrak{o}/\mathfrak{o}^\times} \frac{1}{|\alpha|^{2s}}$$

The ring $\mathbb{Z}[i]$ has multiplicative^[2] maps to the unit circle in \mathbb{C}^\times , namely

$$\chi : \alpha \longrightarrow \left(\frac{\alpha}{|\alpha|} \right)^n$$

For various reasons, we want χ to be invariant by units, that is, $\chi(\eta \cdot \alpha) = \chi(\alpha)$ for units $\eta \in \{\pm 1, \pm i\}$, entailing that χ be of the form $\chi_{4n}(\alpha) = (\alpha/|\alpha|)^{4n}$. With such χ , the corresponding *unramified Hecke L -functions* for $\mathbb{Z}[i]$ are

$$L(s, \chi_{4n}) = \sum_{0 \neq \alpha \in \mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{\chi_{4n}(\alpha)}{|\alpha|^{2s}} = \sum_{0 \neq \alpha \in \mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{(\alpha/|\alpha|)^{4n}}{|\alpha|^{2s}}$$

Meanwhile, the functions $g_n = g_n(\mathbb{Z}[i])$ for this lattice are

$$g_n = \sum_{a, b \in \mathbb{Z}^2 - (0,0)} \frac{1}{(a + bi)^{2n}} = 4 \cdot \sum_{0 \neq \alpha \in \mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{1}{\alpha^{2n}} \quad (\text{vanishing unless } n \in 2\mathbb{Z})$$

Thus,

$$\frac{1}{4} g_{2n} = \sum_{0 \neq \alpha \in \mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{1}{\alpha^{4n}} = \sum_{0 \neq \alpha \in \mathbb{Z}[i]/\mathbb{Z}[i]^\times} \frac{(\alpha/|\alpha|)^{-4n}}{|\alpha|^{2 \cdot 2n}} = L(2n, \chi_{-4n})$$

This is a *special value* of $L(s, \chi_{-4n})$. In this example, $g_3 = 0$, so g_4, g_6, \dots are polynomials in g_2 with *rational* coefficients. That is, the special values $L(4, \chi_{-8}), L(6, \chi_{-12}), \dots$ are polynomials in $L(2, \chi_{-4})$ with rational coefficients.

[2.1.1] **Remark:** A similar discussion applies to lattices $\Lambda = \mathbb{Z} \cdot z + \mathbb{Z}$ where $\mathbb{Z}[z]$ is the ring of algebraic integers in a quadratic extension $\mathbb{Q}(z)$ of \mathbb{Q} .

[2.1.2] **Remark:** The idea is that, just as the special values $\zeta(2n)$ are rational except for appearance of the single transcendental π , the lists of special values $L(2n, \chi)$ need fewer transcendentals than expected.

[2] As usual, a map $\chi : \mathbb{Z}[i] \rightarrow \mathbb{C}^\times$ is *multiplicative* when $\chi(\alpha \cdot \beta) = \chi(\alpha) \cdot \chi(\beta)$ for all $\alpha, \beta \in \mathbb{Z}[i]$.

Bibliography

The examples above are a very tiny and idiosyncratic sample of ideas about *special values* of L -functions. The literature on special values is huge and still growing. For perspective, only through the mid-1970s:

The method of [Riemann 1859] suffices for $\zeta(s)$ itself, and for the Dirichlet L -functions introduced in 1837. Continuing [Blumenthal 1903/4]'s discussion of Hilbert-Blumenthal modular forms, [Hecke 1922/24] conjectured special-values results for Dedekind zeta functions. [Siegel 1937] proved this conjecture, and [Klingen 1961/2] gave a simpler proof using Hilbert-Blumenthal modular forms. [Damerell 1970/71] studied special values of L -functions in a special class including those we discussed above via the Weierstraß equation. Damerell's argument was simplified in [Weil 1976]. [Shimura 1975] and [Shimura 1976] greatly extended investigations of special-values. The results known at the time were put into a very broad conjectural framework by [Deligne 1977/9].

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