

(September 24, 2013)

Liouville's theorem on diophantine approximation

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[0.0.1] Theorem: (*Liouville 1844*) Let $\alpha \in \mathbb{R}$ be an *irrational* algebraic number satisfying $f(\alpha) = 0$ with non-zero irreducible $f \in \mathbb{Z}[x]$ of degree d . Then there is a non-zero constant C such that for every fraction p/q

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{C}{q^d}$$

Proof: By the mean-value theorem, given p/q there is real ξ between α and p/q such that

$$f'(\xi) \left(\alpha - \frac{p}{q} \right) = f(\alpha) - f\left(\frac{p}{q}\right)$$

Since f has integer coefficients and is of degree d , the value $f(p/q)$ is a rational number with denominator at worst q^d . Since f is irreducible, $f(p/q) \neq 0$. Thus, $|f(p/q)| \geq 1/q^d$, and

$$|f'(\xi)| \cdot \left| \alpha - \frac{p}{q} \right| = \left| f(\alpha) - f\left(\frac{p}{q}\right) \right| = \left| 0 - f\left(\frac{p}{q}\right) \right| = \left| f\left(\frac{p}{q}\right) \right| \geq \frac{1}{q^d}$$

Rearranging,

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1/|f'(\xi)|}{q^d}$$

Again since f is irreducible, it does *not* have a double root at α , so $f'(\alpha) \neq 0$. Thus, for ξ sufficiently close to α the derivative $f'(\xi)$ is non-zero. Quantitatively, for sufficiently large q and ξ between α and the best rational approximation p/q to α , $|f'(\xi)| \geq \frac{1}{2} \cdot |f'(\alpha)|$.

Thus, there is q_0 such that for $q \geq q_0$

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{2/|f'(\alpha)|}{q^d}$$

Replace the constant $2/|f'(\alpha)|$ by a smaller constant C , if necessary, so that the same inequality holds for the finitely-many $1 \leq q < q_0$. ///

[0.0.2] Corollary: (*Liouville*) Numbers β *well approximable* by rational numbers, in the sense that, for every $d \geq 1$ and for every positive constant C , there is a rational p/q such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{C}{q^d}$$

are *transcendental*, that is, not algebraic, over \mathbb{Q} . ///

[0.0.3] Example: The real number

$$\beta = \sum_{n \geq 1} \frac{1}{2^{n!}} \quad (\text{exponent is } n!)$$

is transcendental, because there is a rational approximation

$$\left| \beta - \sum_{n \leq N} \frac{1}{2^{n!}} \right| = \sum_{n > N} \frac{1}{2^{n!}} < \frac{2}{2^{(N+1)!}} = \frac{1}{2^{(N+1)!-1}}$$

with

$$\sum_{n \leq N} \frac{1}{2^{n!}} = \frac{\sum_{n \leq N} 2^{N!-n!}}{2^{N!}} = \frac{\text{integer}}{2^{N!}}$$

The ratio $\frac{(N+1)!-1}{N!}$ is unbounded as $N \rightarrow +\infty$, so β is *well-approximable* by rationals. ///

[0.0.4] **Remark:** For numbers α *not* well approximable by rational numbers, the *equidistribution* of the sequence $\ell \cdot \alpha$ is quantifiable in terms of Weyl's criterion. That is, $|\alpha - \frac{p}{q}| \gg \frac{1}{q^d}$ gives

$$|n\alpha - m| \gg |n| \cdot \left| \alpha - \frac{m}{n} \right| \gg |n| \cdot |n|^{-d} \quad (\text{for all integers } m \text{ and } n \neq 0)$$

giving

$$|1 - e^{2\pi i n \alpha}| \gg \frac{1}{n^{d-1}} \quad (\text{implied constant uniform in } n \neq 0)$$

Thus, in the Weyl criterion, we have an estimate *uniform* in Fourier index n :

$$\left| \frac{1}{N} \sum_{\ell=1}^N e^{2\pi i n \cdot \ell \alpha} \right| \leq \frac{1}{N} \cdot \frac{2}{|1 - e^{2\pi i n \alpha}|} \ll \frac{1}{N} \cdot n^{d-1} \quad (\text{uniformly in } n \neq 0)$$

[0.0.5] **Remark:** The *Thue-Siegel-Roth* improves the exponent in the lower bound for the estimate of error in approximating an irrational algebraic number α by rationals. Specifically, for every $\varepsilon > 0$, Roth proved that there are only *finitely-many* fractions p/q satisfying

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}$$

This had been conjectured by Siegel in 1921. Thue, Siegel, and Dyson had successively improved Liouville's original exponent d , until Roth proved Siegel's conjectured exponent in 1955, and won a Fields Medal for this work.

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