(*September 24, 2013*)

Liouville's theorem on diophantine approximation

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/mfms/notes_2013-14/04b_Liouville_approx.pdf]

[0.0.1] Theorem: *(Liouville 1844)* Let $\alpha \in \mathbb{R}$ be an *irrational* algebraic number satisfying $f(\alpha) = 0$ with non-zero irreducible $f \in \mathbb{Z}[x]$ of degree d. Then there is a non-zero constant C such that for every fraction p/q

$$\left|\alpha - \frac{p}{q}\right| \;\geq\; \frac{C}{q^d}$$

Proof: By the mean-value theorem, given p/q there is real ξ between α and p/q such that

$$f'(\xi)\big(\alpha - \frac{p}{q}\big) \;=\; f(\alpha) - f\big(\frac{p}{q}\big)$$

Since f has integer coefficients and is of degree d, the value $f(p/q)$ is a rational number with denominator at worst q^d. Since f is irreducible, $f(p/q) \neq 0$. Thus, $|f(p/q)| \geq 1/q^d$, and

$$|f'(\xi)| \cdot \left|\alpha - \frac{p}{q}\right| \;=\; \left|f(\alpha) - f\big(\frac{p}{q}\big)\right| \;=\; \left|0 - f\big(\frac{p}{q}\big)\right| \;=\; \left|f\big(\frac{p}{q}\big)\right| \;\geq\; \frac{1}{q^d}$$

Rearranging,

$$\left|\alpha - \frac{p}{q}\right| \;\geq\; \frac{1/|f'(\xi)|}{q^d}$$

Again since f is irreducible, it does *not* have a double root at α, so $f'(\alpha) \neq 0$. Thus, for ξ sufficiently close to α the derivative $f'(\xi)$ is non-zero. Quantitatively, for sufficiently large q and ξ between α and the best rational approximation p/q to α, $|f'(\xi)| \geq \frac{1}{2} \cdot |f'(\alpha)|$.

Thus, there is q_o such that for $q \geq q_o$

$$\left|\alpha - \frac{p}{q}\right| \;\geq\; \frac{2/|f'(\alpha)|}{q^d}$$

Replace the constant $2/|f'(\alpha)|$ by a smaller constant C, if necessary, so that the same inequality holds for the finitely-many $1 \leq q < q_o$. ///

[0.0.2] Corollary: *(Liouville)* Numbers β *well approximable* by rational numbers, in the sense that, for every $d \geq 1$ and for every positive constant C, there is a rational p/q such that

$$\left|\alpha - \frac{p}{q}\right| \;<\; \frac{C}{q^d}$$

are *transcendental*, that is, not algebraic, over $\mathbb{Q}$. ///

[0.0.3] Example: The real number

$$\beta \;=\; \sum_{n\geq 1} \frac{1}{2^{n!}} \qquad\qquad \text{(exponent is } n!\text{)}$$

is transcendental, because there is a rational approximation

$$\left|\beta - \sum_{n\leq N} \frac{1}{2^{n!}}\right| \;=\; \sum_{n>N} \frac{1}{2^{n!}} \;<\; \frac{2}{2^{(N+1)!}} \;=\; \frac{1}{2^{(N+1)!-1}}$$

with

$$\sum_{n\le N} \frac{1}{2^{n!}} \;=\; \frac{\sum_{n\le N} 2^{N!-n!}}{2^{N!}} \;=\; \frac{\text{integer}}{2^{N!}}$$

The ratio $\frac{(N+1)!-1}{N!}$ is unbounded as $N \to +\infty$, so β is *well-approximable* by rationals. ///

[0.0.4] Remark: For numbers α *not* well approximable by rational numbers, the *equidistribution* of the sequence $\ell \cdot \alpha$ is quantifiable in terms of Weyl's criterion. That is, $|\alpha - \frac{p}{q}| \gg \frac{1}{q^d}$ gives

$$|n\alpha - m| \;\gg\; |n| \cdot \left|\alpha - \frac{m}{n}\right| \;\gg\; |n| \cdot |n|^{-d} \qquad \text{(for all integers } m \text{ and } n \neq 0)$$

giving

$$|1 - e^{2\pi i n\alpha}| \gg \frac{1}{n^{d-1}} \qquad \text{(implied constant uniform in } n \neq 0)$$

Thus, in the Weyl criterion, we have an estimate *uniform* in Fourier index n:

$$\left|\frac{1}{N}\sum_{\ell=1}^{N} e^{2\pi i n\cdot \ell\alpha}\right| \;\le\; \frac{1}{N} \cdot \frac{2}{|1 - e^{2\pi i n\alpha}|} \;\ll\; \frac{1}{N} \cdot n^{d-1} \qquad \text{(uniformly in } n \neq 0)$$

[0.0.5] Remark: The *Thue-Siegel-Roth* improves the exponent in the lower bound for the estimate of error in approximating an irrational algebraic number α by rationals. Specifically, for every $\varepsilon > 0$, Roth proved that there are only *finitely-many* fractions p/q satisfying

$$\left|\alpha - \frac{p}{q}\right| \;<\; \frac{1}{q^{2+\varepsilon}}$$

This had been conjectured by Siegel in 1921. Thue, Siegel, and Dyson had successively improved Liouville's original exponent d, until Roth proved Siegel's conjectured exponent in 1955, and won a Fields Medal for this work.

Bibliography

[Dyson 1947] F.J. Dyson, *The approximation to algebraic numbers by rationals*, Acta Math. **79** (1947), 225-240.

[Liouville 1851] J. Liouville, *Sur des classes très-étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationalles algébriques*, J. Math. pures et app. **16** (1851), 133-142.

[Roth 1955] K. F. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), 1-20.

[Siegel 1921] C. L. Siegel, *Approximation algebraischer Zahlen*, Math. Zeit. **10** (1921), 173-213.

[Thue 1909] A. Thue, *Über Annäherungswerte algebraischer Zahlen*, J. reine angew. Math. **135** (1909), 284-305.
