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First proof of $L(1, \chi) \neq 0$

Paul Garrett garrett@math.umn.edu <http://www.math.umn.edu/~garrett/>

[This document is http://www.math.umn.edu/~garrett/m/mfms/notes_2013-14/05b_ugly_nonvanishing.pdf]

1. First proof of non-vanishing on $\operatorname{Re}(s) = 1$
2. Landau's lemma: Dirichlet series with positive coefficients

The subtle element in Dirichlet's theorem about primes in arithmetic progressions is *non-vanishing of L -functions* $L(s, \chi)$ at $s = 1$.

Here, we give a rather ugly and unexplanatory proof. However, the argument has few prerequisites. It uses Landau's Lemma, which we prove here.

1. First proof of non-vanishing on $\operatorname{Re}(s) = 1$

We prove that $L(1, \chi) \neq 0$ for all $\chi \bmod N$, *granting* that these L -functions have meromorphic extensions to some neighborhood of $s = 1$. We also need to know that for the trivial character $\chi_o \bmod N$ the L -function $L(s, \chi_o)$ has a *simple* pole at $s = 1$.

[1.0.1] **Theorem:** For a Dirichlet character $\chi \bmod N$ other than the trivial character $\chi_o \bmod N$,

$$L(1, \chi) \neq 0$$

Proof: To prove that the L -functions $L(s, \chi)$ do not vanish at $s = 1$, and in fact do not vanish on the whole line^[1] $\operatorname{Re}(s) = 1$, direct arguments involve tricks similar to what we do here.

First, for χ whose square is *not* the trivial character χ_o modulo N , the standard trick is to consider

$$\lambda(s) = L(s, \chi_o)^3 \cdot L(s, \chi)^4 \cdot L(s, \chi^2)$$

Then, letting $\sigma = \operatorname{Re}(s)$, from the Euler product expressions for the L -functions noted earlier, in the region of convergence,

$$|\lambda(s)| = \left| \exp \left(\sum_{m,p} \frac{3 + 4\chi(p^m) + \chi^2(p^m)}{mp^{m\sigma}} \right) \right| = \exp \left| \sum_{m,p} \frac{3 + 4 \cos \theta_{m,p} + \cos 2\theta_{m,p}}{mp^{m\sigma}} \right|$$

where for each m and p we let

$$\theta_{m,p} = (\text{the argument of } \chi(p^m)) \in \mathbb{R}$$

The trick, presumably found after considerable experimentation, is that for *any* real θ

$$3 + 4 \cos \theta + \cos 2\theta = 3 + 4 \cos \theta + 2 \cos^2 \theta - 1 = 2 + 4 \cos \theta + 2 \cos^2 \theta = 2(1 + \cos \theta)^2 \geq 0$$

[1] Non-vanishing of $\zeta(s)$ on the whole line $\operatorname{Re}(s) = 1$ yields the Prime Number Theorem: let $\pi(x)$ be the number of primes less than x . Then $\pi(x) \sim x/\ln x$, meaning that the limit of the ratio of the two sides as $x \rightarrow \infty$ is 1. This was first proven in 1896, separately, by Hadamard and de la Vallée Poussin. The same sort of argument also gives an analogous *asymptotic* statement about primes in each congruence class modulo N , namely that $\pi_{a,N}(x) \sim x/[\varphi(N) \cdot \ln x]$, where $\gcd(a, N) = 1$ and φ is Euler's totient function.

Therefore, miraculously, all the terms inside the large sum being exponentiated are non-negative, and

$$|\lambda(s)| \geq e^0 = 1$$

In particular, if $L(1, \chi) = 0$ were to be 0, then, since $L(s, \chi_o)$ has a simple pole at $s = 1$ and since $L(s, \chi^2)$ does *not* have a pole (since $\chi^2 \neq \chi_o$), the multiplicity ≥ 4 of the 0 in the product of L -functions would overwhelm the three-fold pole, and $\lambda(1) = 0$. This would contradict the inequality just obtained.

For $\chi^2 = \chi_o$, instead consider

$$\lambda(s) = L(s, \chi) \cdot L(s, \chi_o) = \exp\left(\sum_{p,m} \frac{1 + \chi(p^m)}{mp^{ms}}\right)$$

If $L(1, \chi) = 0$, then this would cancel the simple pole of $L(s, \chi_o)$ at 1, giving a non-zero finite value at $s = 1$. The series inside the exponentiation is a *Dirichlet series with non-negative coefficients*, and for real s

$$\sum_{p,m} \frac{1 + \chi(p^m)}{mp^{ms}} \geq \sum_{p,m \text{ even}} \frac{1+1}{mp^{ms}} = \sum_{p,m} \frac{1+1}{2mp^{2ms}} = \sum_{p,m} \frac{1}{mp^{2ms}} = \log \zeta(2s)$$

Since $\zeta(2s)$ has a simple pole at $s = \frac{1}{2}$ the series

$$\log(L(s, \chi) \cdot L(s, \chi_o)) = \sum_{p,m} \frac{1 + \chi(p^m)}{mp^{ms}} \geq \log \zeta(2s)$$

necessarily blows up as $s \rightarrow \frac{1}{2}^+$. But by *Landau's Lemma* below, a Dirichlet series with non-negative coefficients cannot blow up as $s \rightarrow s_o$ along the real line unless the function represented by the series fails to be holomorphic at s_o . Since the function given by $\lambda(s)$ is holomorphic at $s = 1/2$, this gives a contradiction to the supposition that $\lambda(s)$ is holomorphic at $s = 1$ (which had allowed this discussion at $s = 1/2$). That is, $L(1, \chi) \neq 0$. ///

[1.0.2] Remark: Again, the above argument is quick, but unilluminating. We will give better proofs later.

2. Landau's Lemma on Dirichlet series with positive coefficients

[2.0.1] Theorem: (*Landau*) Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series with real coefficients $a_n \geq 0$. Suppose that the series defining $f(s)$ converges for $\text{Re}(s) > \sigma_o$. Suppose further that the function f extends to a function holomorphic in a neighborhood of $s = \sigma_o$. Then, in fact, the series defining $f(s)$ converges for $\text{Re}(s) > \sigma_o - \varepsilon$ for some $\varepsilon > 0$.

Proof: First, by replacing s by $s - \sigma_o$ reduce to the case that $\sigma_o = 0$. Since the function $f(s)$ given by the series is holomorphic on $\text{Re}(s) > 0$ and on a neighborhood of 0, there is $\varepsilon > 0$ such that $f(s)$ is holomorphic on $|s - 1| < 1 + 2\varepsilon$, and the power series for the function converges nicely on this open disk. Differentiating the original series termwise (Abel's theorem), evaluate the derivatives of $f(s)$ at $s = 1$ as

$$f^{(i)}(1) = \sum_n \frac{(-\log n)^i a_n}{n} = (-1)^i \sum_n \frac{(\log n)^i a_n}{n}$$

and Cauchy's formulas yield, for $|s - 1| < 1 + 2\varepsilon$,

$$f(s) = \sum_{i \geq 0} \frac{f^{(i)}(1)}{i!} (s - 1)^i$$

In particular, for $s = -\varepsilon$, we are assured of the convergence to $f(-\varepsilon)$ of

$$f(-\varepsilon) = \sum_{i \geq 0} \frac{f^{(i)}(1)}{i!} (-\varepsilon - 1)^i$$

Since $(-1)^i f^{(i)}(1)$ is a *positive* Dirichlet series, move the powers of -1 a little to obtain

$$f(-\varepsilon) = \sum_{i \geq 0} \frac{(-1)^i f^{(i)}(1)}{i!} (\varepsilon + 1)^i$$

The series

$$(-1)^i f^{(i)}(1) = \sum_n (\log n)^i \frac{a_n}{n}$$

has positive terms, so the double series, convergent, with positive terms,

$$f(-\varepsilon) = \sum_{n,i} \frac{a_n (\log n)^i}{i!} (1 + \varepsilon)^i \frac{1}{n}$$

can be rearranged to

$$f(-\varepsilon) = \sum_n \frac{a_n}{n} \left(\sum_i \frac{(\log n)^i (1 + \varepsilon)^i}{i!} \right) = \sum_n \frac{a_n}{n} n^{(1+\varepsilon)} = \sum_n \frac{a_n}{n^{-\varepsilon}}$$

That is, the latter series converges (absolutely).

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