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Harmonic theta series and equidistribution

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The first (slightly frivolous) application here is to obtain a formula for the number of ways to express a positive integer as a sum of squares of eight integers. For example, we will prove

$$\text{number of ways to express odd prime } p \text{ as sum of 8 squares} = 16(p^3 + 1)$$

This will follow from the fact that the theta series

$$\theta(z) = \sum_{v \in \mathbb{Z}^8} e^{\pi i |v|^2 z}$$

is a holomorphic modular form of weight 4 for the subgroup^[1] Γ_θ , from the fact that there are no weight 4 cuspforms for Γ_θ , and from explicit computation of the Fourier coefficients of the two types of weight 4 Eisenstein series for Γ_θ .

Toward a less frivolous result, recall that the *irreducibility* of spaces H_d of homogeneous, degree d harmonic polynomials f on \mathbb{R}^n as $O(n)$ -spaces gives Hecke's identity

$$\left(f(x) e^{-\pi |x|^2} \right)^\sim(\xi) = i^{-d} f(\xi) e^{-\pi |\xi|^2}$$

As below, this proves that the *harmonic theta series*

$$\theta_f(z) = \sum_{v \in \mathbb{Z}^{8n}} f(v) e^{\pi i |v|^2 z} \quad (f \text{ harmonic, homogeneous degree } d, \text{ on } \mathbb{R}^{8n})$$

is a holomorphic modular form of weight $d + 4n$ for Γ_θ , Taking dimension divisible by 8 avoids several difficulties, some of them serious.^[2] For degree $d > 0$, in fact these prove to be *cuspforms*. Thus, a Hecke-style estimate applies to Fourier coefficients:

$$N^{\text{th}} \text{ Fourier coefficient} = O(N^{\frac{d+4n}{2}}) \quad (\text{for } d > 0)$$

[1] Recall that

$$\Gamma_\theta = \Gamma(2) \cup w\Gamma(2) \quad (\text{with } \Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \right\})$$

and w is the long Weyl element $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

[2] The weight is half the dimension, and Eisenstein series only converge in dimensions > 2 , so dimension at least 5 is needed for convergence of Eisenstein series. *Even* dimension is needed to avoid a more technically difficult type of modular forms, of *half-integral weight*. Dimension divisible by 4, or by 8, gives the simplest modular forms for Γ_θ rather than a slightly more complicated type.

For $d = 0$, the theta series has a weight- $4n$ Eisenstein series component, so, again using the Hecke estimate on the remaining cuspidal component, its Fourier N^{th} series have an asymptotic lower bound $\geq N^{4n-1}$.

This sets up an application of Weyl's equidistribution criterion, to the sequence of projections to the unit sphere $S^{8n-1} \subset \mathbb{R}^{8n}$ of the sets

$$X_N = \{v \in \mathbb{Z}^{8n} : |v|^2 = N\}$$

as $N \rightarrow +\infty$. Weyl's criterion requires that we show that

$$\lim_{N \rightarrow \infty} \frac{1}{\#X_N} \sum_{v \in X_N} f\left(\frac{v}{|v|}\right) = 0 \quad (\text{for } f \in H_d \text{ for } d > 0)$$

A natural generating function for summing $f(v/|v|)$ over X_N is

$$\sum_{0 \leq N \in \mathbb{Z}} \sum_{v \in \mathbb{Z}^{8n} : |v|^2 = N} f(v) e^{\pi i N z} = \sum_{v \in \mathbb{Z}^{8n}} f(v) e^{\pi i |v|^2 z} = \theta_f(z)$$

Hecke's estimate that the N^{th} coefficient is $O(N^{\frac{d+4n}{2}})$ gives, for f homogeneous of degree d ,

$$\left| \sum_{v \in X_N} f\left(\frac{v}{|v|}\right) \right| = \frac{1}{\sqrt{N}^d} \left| \sum_{v \in X_N} f(v) \right| \ll \frac{1}{\sqrt{N}^d} N^{\frac{d+4n}{2}} \leq N^{2n}$$

To satisfy Weyl's criterion, given the asymptotic lower bound for $\#X_N$ from the Eisenstein series, it suffices that

$$\lim_{N \rightarrow \infty} \frac{1}{N^{4n-1}} \cdot N^{2n} = 0$$

which holds for $4n - 1 > 2n$, which is $n > \frac{1}{2}$. The dimension is $8n$, so Hecke's estimate gives a sufficient bound for Weyl's criterion for dimension > 4 . [3]

1. Holomorphic modular forms for Γ_θ

[1.1] Standard fundamental domain

We proved that the translation $z \rightarrow z + 2$ and the inversion $z \rightarrow -1/z$ generate Γ_θ , and there is the standard fundamental domain

$$F = \{z \in \mathfrak{H} : |z| \geq 1, -1 \leq z \leq 1\}$$

There are two Γ_θ -inequivalent cusps at the boundary of the standard fundamental domain, $i\infty$ and 1 , since -1 is identified with $+1$ by $z \rightarrow z + 2$.

[1.2] No odd-weight modular forms for Γ_θ

Just as for $SL_2(\mathbb{Z})$, there are no (not-identically-zero) odd-weight holomorphic modular forms f for Γ_θ , since $-1_2 \in \Gamma_\theta$, and

$$f(z) = f \left| \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} (z) = (-1)^k \cdot f(z)$$

[3] Deligne's estimate $O(N^{\frac{d+4n-1}{2}})$ gives an estimate meeting Weyl's criterion for $4n-1 > 2n - \frac{1}{2}$, which is $n > \frac{1}{4}$, with dimension $8n$, so Deligne would let us take dimension > 2 . This includes even dimension 4, but the Eisenstein series of weight $4/2 = 2$ do not converge, so there are other complications. And, again, odd dimension give half-integral weight modular forms, which have their own complications, even more severe.

[1.3] Fourier expansion at $i\infty$

Similar to $SL_2(\mathbb{Z})$, because of invariance under the translation $z \rightarrow z + 2$, holomorphic modular forms for Γ_θ have Fourier series expansions of the form

$$f(z) = \sum_{n \geq \mathbb{Z}} c_n e^{\pi i n z}$$

with exponentials $e^{\pi i n z}$ rather than $e^{2\pi i n z}$ as for $SL_2(\mathbb{Z})$, since the latter contains the translation $z \rightarrow z + 1$. That is, the *width* of the cusp $i\infty$ of Γ_θ is 2, not 1, as it was for $SL_2(\mathbb{Z})$.

Require that f is bounded as $y \rightarrow +\infty$. The Fourier coefficients are obtained by the expected formula, noting that the exponential involving y stays with the constant:

$$\begin{aligned} \frac{1}{2} \int_{-1}^1 e^{-\pi i n x} f(x + iy) dx &= \sum_m c_m \cdot \frac{1}{2} \int_{-1}^1 e^{-\pi i n x} e^{\pi i m(x+iy)} dx \\ &= \sum_m c_m \cdot \frac{1}{2} \int_{-1}^1 e^{-\pi i n x} e^{\pi i m(x+iy)} dx = c_n \cdot e^{-2\pi n y} \end{aligned}$$

Thus, the boundedness gives

$$|c_n e^{-2\pi n y}| = \left| \frac{1}{2} \int_{-1}^1 e^{-\pi i n x} f(x + iy) dx \right| \leq \int_{-1}^1 |f(x + iy)| dx \ll 1$$

For $n < 0$, letting $y \rightarrow +\infty$ proves $c_n = 0$. The *order of vanishing* of f at $i\infty$ is the lowest index n_o such that $c_{n_o} \neq 0$. This much is parallel to $SL_2(\mathbb{Z})$.

[1.4] Boundedness at the other cusp 1 The details of the proof of the *divisor formula* below explain, at least with hindsight, that part of the definition of weight $2k$ *holomorphic modular form* f for Γ_θ require that, for $g \in SL_2(\mathbb{Z})$ with $g(i\infty) = 1$, as $y \rightarrow +\infty$ the value $(f|_{2k}g)(z)$ is bounded.

[1.4.1] Remark: That is, as it happens, requiring that $f(z)$ itself be bounded as $z \rightarrow 1$ inside the standard fundamental region is *too strong* a condition. By accident, it would exclude some holomorphic Eisenstein series needed for a coherent discussion.

The action of $SL_2(\mathbb{C})$ on complex projective space $\mathbb{C} \cup \{\infty\}$ maps lines-and-circles to lines-and-circles, so any $g \in SL_2(\mathbb{R})$ mapping $i\infty$ to 1 maps regions $\{y > T\}$ to open disks tangent to the real line at 1. For fixed g , as T increases, the circles shrink. Thus, in this context, *approaching the cusp 1* means to approach within such shrinking circles. In particular, this *excludes* tangential approach, or any approach other than asymptotically vertical.

[1.5] Fourier expansion at the other cusp 1 Also from the details of the *divisor formula* the *order of vanishing* of f at the cusp 1 should be expressed in terms of a *Fourier expansion* of f at 1. For this, we need to change coordinates on \mathfrak{H} so that the isotropy group of 1 in Γ_θ becomes ordinary translations. That is, we want $g \in SL_2(\mathbb{Z})$ to map 1 to $i\infty$, since for $\gamma(1) = 1$,

$$\gamma g^{-1}(i\infty) = \gamma(1) = 1 = g^{-1}(i\infty)$$

gives $(g\gamma g^{-1})(i\infty) = i\infty$. The isotropy group of $i\infty$ in $SL_2(\mathbb{R})$ is upper-triangular matrices P , so

$$g(\text{isotropy group of 1 in } \Gamma_\theta)g^{-1} \subset P$$

and, equivalently,

$$\text{isotropy group of 1 in } \Gamma_\theta = \Gamma_\theta \cap g^{-1}Pg$$

An explicit choice of mapping 1 to $i\infty$ is easily accomplished by translating and inverting:

$$1 \longrightarrow \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} (1) = 0 \longrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (0) = i\infty$$

That is, $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ maps $1 \rightarrow i\infty$. The *translations* in the isotropy group of 1 are $\Gamma_\theta \cap g^{-1}Ng$, where $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is the group of translations in $SL_2(\mathbb{R})$:

$$\text{translations in isotropy group of } 1 = \Gamma_\theta \cap \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -b+1 & b \\ -b & b+1 \end{pmatrix}$$

The latter is in Γ_θ if and only if $b \in \mathbb{Z}$. That is, the *width* of the cusp 1 of Γ_θ is 1, in contrast to the cusp $i\infty$ which has width 2 for Γ_θ .

The Fourier expansion at 1 of a modular form f for Γ_θ is really the Fourier expansion of $(f|_{2k}g)(z)$ for $g \in SL_2(\mathbb{Z})$ with $g(i\infty) = 1$. The determination that the *width* is 1 shows that the expansion will be of the form

$$(f|_{2k}g)(z) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi inz}$$

The same argument as for $i\infty$ shows that boundedness implies that the negative-index Fourier coefficients are all 0.

[1.6] Hecke's estimate on Fourier coefficients of cuspforms Just as for $SL_2(\mathbb{Z})$, it is straightforward to obtain a useful asymptotic bound on Fourier coefficients of holomorphic cuspforms for Γ_θ . This will be used a little frivolously in obtaining an asymptotic for the number of ways to represent integers as sums of squares, and less frivolously in proving some equidistribution results on spheres.

[1.6.1] Theorem: (*Hecke*) For $f(z) = \sum_{n \geq 1} c_n e^{\pi inz}$ a holomorphic cuspform of weight $2k$ for Γ_θ ,

$$c_m = O(n^{\frac{2k}{2}})$$

Proof: Since f is bounded as $y \rightarrow +\infty$,

$$|c_n \cdot e^{-\pi ny}| = \frac{1}{2} \left| \int_0^2 e^{-\pi ix} f(x+iy) dx \right| \leq \frac{1}{2} \int_0^2 |f(x+iy)| dx \ll_f 1$$

This gives a bad preliminary estimate $|c_n| \ll_f e^{\pi n}$, by taking $y = 1$. As for $SL_2(\mathbb{Z})$, this gives exponential decay of $f(x+iy)$ as $y \rightarrow +\infty$:

$$|f(x+iy)| \leq \sum_{n \geq 1} |c_n| e^{-\pi ny} \ll_f \sum_{n \geq 1} e^{\pi n} e^{-\pi ny} = \frac{e^{-\pi(y-1)}}{1 - e^{\pi(y-1)}}$$

A nearly identical discussion applies to f near the cusp 1, that is, with $g \in SL_2(\mathbb{Z})$ such that $g(i\infty) = 1$, the same discussion applies to $f|_{2k}g$ near $i\infty$. Namely,

$$(f|_{2k}g)(x+iy) \ll_f e^{-2\pi(y-1)} \quad (\text{for } y \gg 1)$$

For example, with

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

$$(f|_{2k}g)(x+iy) = z^{-2k} \cdot f(gz) \ll_f e^{-2\pi(y-1)} \ll_f e^{-2\pi(y-1)} \quad (\text{for } y \gg 2)$$

Since the cusp 1 has width 1, it suffices to consider $0 \leq x \leq 1$, and in the region $0 \leq x \leq 1$ and $y \geq 2$,

$$|y| \ll |z| \ll |y|$$

so

$$f(g(x+iy)) \ll y^{2k} \cdot |f(gz)| \ll_f y^{2k} e^{-2\pi(y-1)} \quad (\text{in } 0 \leq x \leq 1 \text{ and } y \geq 2)$$

The relevant qualitative point is that this is *of rapid decay* as $y \rightarrow \infty$. That is, $f(z)$ is of rapid decay as z approaches either cusp within a fixed fundamental domain.

Since f is continuous, $y^{\frac{2k}{2}}|f(z)|$ is *bounded* in the fundamental domain. It is also Γ_θ -invariant, so is bounded on the whole \mathfrak{H} .

Thus, as with $SL_2(\mathbb{Z})$, we boot-strap ourselves:

$$|c_n \cdot e^{-\pi n y} \cdot y^{\frac{2k}{2}}| = \frac{1}{2} y^{\frac{2k}{2}} \left| \int_0^2 e^{-\pi i n x} f(x+iy) dx \right| \leq \frac{1}{2} \int_0^2 y^k |f(x+iy)| dx \ll_f 1$$

From $|c_n| \ll_f y^{-\frac{2k}{2}} e^{\pi n y}$ for all $y > 0$, optimize choice of y to minimize the upper bound: solve

$$0 = \frac{\partial}{\partial y} y^{-k} e^{\pi n y} = -k y^{-k-1} e^{\pi n y} + \pi n y^{-k} e^{\pi n y} = (-k + \pi n y) \cdot y^{-k-1} e^{\pi n y}$$

Thus, $y = k/\pi n$, giving

$$|c_n| \ll_f \left(\frac{k}{\pi n} \right)^{-\frac{2k}{2}} e^{\pi n \cdot k/\pi n} \ll_{f,k} n^{\frac{2k}{k}}$$

which is Hecke's estimate. ///

[1.7] Ramanujan's Delta In the discussion of $SL_2(\mathbb{Z})$, we saw that Ramanujan's cuspform $\Delta(z)$ of weight 12 does not vanish on \mathfrak{H} , but only at $i\infty$. It is certainly also a cuspform for Γ_θ . Since the *width* of the cusp 1 is just 1, $\Delta(z)$ vanishes to order 1 at the cusp 1. However, the width of the cusp $i\infty$ for Γ_θ is 2, so in the appropriate coordinates for Γ_θ , $\Delta(z)$ vanishes to order 2 at $i\infty$.

Apart from this possible surprise, there is the consequence that it is no longer possible to divide general cuspforms for Γ_θ by $\Delta(z)$ and obtain holomorphic modular forms, unless the given cuspform happens to vanish to order 2 at $i\infty$.

In discussion of the *divisor formula* below, we will discover that there *is* an essentially unique cuspform of weight 8, vanishing to first order at both cusps, and not vanishing at any point of \mathfrak{H} .

2. Holomorphic Eisenstein series for Γ_θ

One use of holomorphic Eisenstein series is to subtract them from a given holomorphic modular form to produce a cuspform. That is, referring to the two (equivalence classes of) cusps by their representatives $i\infty$ and 1, for each weight $4 \leq 2k \in 2\mathbb{Z}$, we want two Eisenstein series $E_{2k}^{(i\infty)}$ and E_{2k}^1 so that, with $g \in SL_2(\mathbb{Z})$ such that $g(i\infty) = 1$,

$$\begin{cases} E_{2k}^{(i\infty)}(i\infty) & = 1 \\ E_{2k}^{(i\infty)}|_{2k}g(1) & = 0 \end{cases} \quad \begin{cases} E_{2k}^1(i\infty) & = 0 \\ E_{2k}^1|_{2k}g(1) & = 1 \end{cases}$$

[2.0.1] Remark: That is, roughly, the Eisenstein series attached to a cusp σ should essentially have value 1 there and should essentially have value 0 at the other cusp. This does not quite literally happen, except at $i\infty$. Rather, the weight $2k$ action intervenes, reducing evaluation at other cusps to evaluation at $i\infty$.

[2.1] **Moving Eisenstein series around** For this subsection, we look at Eisenstein series defined by *congruence conditions* modulo N , because the patterns are clearer than if we'd just treat $N = 2$.

Fix $N \geq 1$, fix weight $4 \leq 2k \in 2\mathbb{Z}$, and for $c_o, d_o \bmod N$ define a slightly different type of Eisenstein series

$$\tilde{E}_{(c_o, d_o)}(z) = \sum_{(0,0) \neq (c,d), (c,d) = (c_o, d_o) \bmod N} \frac{1}{(cz + d)^{2k}} \quad (\text{with } (c, d) \in \mathbb{Z}^2)$$

Generally, such an Eisenstein series will be a modular form only for the principal congruence subgroup

$$\Gamma(N) = \Gamma_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$$

One virtue of this description is the simplicity of behavior under the weight $2k$ action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in the full group $SL_2(\mathbb{Z})$:

$$\begin{aligned} \tilde{E}_{(c_o, d_o)}|_{2k}g(z) &= (cz + d)^{-2k} \sum_{(0,0) \neq (m,n), (m,n) = (c_o, d_o) \bmod N} \frac{1}{\left(m \frac{az+b}{cz+d} + n\right)^{2k}} = \\ &= \sum_{(m,n) = (c_o, d_o) \bmod N} \frac{1}{(m(az + b) + n(cz + d))^{2k}} = \sum_{(m,n) = (c_o, d_o) \bmod N} \frac{1}{((ma + nc)z + (mb + nd))^{2k}} \end{aligned}$$

Mod N we have

$$(ma + nc \quad mb + nd) = (m \quad n) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (c_o \quad d_o) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod N$$

This gives the useful relation

$$\tilde{E}_{(c_o, d_o)}|_{2k}g = \tilde{E}_{(c_o, d_o)g}$$

In particular, the Fourier expansions of these Eisenstein series at various cusps are systematically accessible, as below. A more systematic version of the above description, for fixed level N and weight $2k$, is to let φ be a \mathbb{C} -valued function on $(\mathbb{Z}/N)^2$, and put

$$\tilde{E}_\varphi(z) = \sum_{(0,0) \neq (c,d) \in \mathbb{Z}^2} \varphi(c, d) \frac{1}{(cz + d)^{2k}}$$

The same computation shows that, for $g \in SL_2(\mathbb{Z})$,

$$\tilde{E}_\varphi|_{2k}g = \tilde{E}_{\varphi \circ g^{-1}}$$

In effect, the previous version \tilde{E}_{c_o, d_o} used the function

$$\varphi(c, d) = \begin{cases} 1 & (\text{when } (c, d) = (c_o, d_o) \bmod N) \\ 0 & (\text{otherwise}) \end{cases}$$

The form \tilde{E}_φ is readily expressed in terms of the \tilde{E}_{c_o, d_o} 's, by

$$\begin{aligned} \tilde{E}_\varphi(z) &= \sum_{(0,0) \neq (c,d) \in \mathbb{Z}^2} \varphi(c, d) \frac{1}{(cz + d)^{2k}} = \sum_{(c_o, d_o) \bmod N} \varphi(c_o, d_o) \sum_{0 \neq (c,d) = (c_o, d_o) \bmod N} \frac{1}{(cz + d)^{2k}} \\ &= \sum_{(c_o, d_o) \bmod N} \varphi(c_o, d_o) \cdot \tilde{E}_{c_o, d_o}(z) \end{aligned}$$

[2.2] Fourier expansions of Eisenstein series Description of Eisenstein series \tilde{E} as sums *with* congruence conditions but *without* the coprimality condition, as just above, facilitates determination of Fourier expansions.

Break the sum expressing the weight $2k$ level N Eisenstein series \tilde{E}_φ into subsums invariant under the translation $z \rightarrow z + N$ in $\Gamma(N)$:

$$\tilde{E}_\varphi(z) = \sum_{0 \neq (c,d)} \frac{\varphi(c,d)}{(cz+d)^{2k}} = \sum_{0 \neq d} \frac{\varphi(0,d)}{d^{2k}} + \sum_{0 \neq c} \sum_{d \in \mathbb{Z}} \frac{\varphi(c,d)}{(cz+d)^{2k}} = \sum_{0 \neq d} \frac{\varphi(0,d)}{d^{2k}} + \sum_{0 \neq c} \frac{1}{c^{2k}} \sum_{d \in \mathbb{Z}} \frac{\varphi(c,d)}{(z + \frac{d}{c})^{2k}}$$

The first of the latter two sums, being a constant, makes a contribution to the 0^{th} Fourier coefficient of \tilde{E}_{c_0, d_0} . As with the level one holomorphic Eisenstein series, we will see that the rest of the sum does not contribute to the 0^{th} Fourier coefficient, that is, we anticipate that

$$\tilde{E}_\varphi(i\infty) = \sum_{0 \neq d} \frac{\varphi(0,d)}{d^{2k}}$$

For each $c \neq 0$, the inner sum over d can be rewritten by letting $d = d_1 + Ncl$ with $d_1 \in \mathbb{Z}/Nc$. Since $\varphi(c, Ncl + d_1)$ does not depend on l ,

$$\sum_{d \in \mathbb{Z}} \frac{\varphi(c,d)}{(z + \frac{d}{c})^{2k}} = \sum_{d_1 \in \mathbb{Z}/Nc} \sum_{l \in \mathbb{Z}} \frac{\varphi(c, Ncl + d_1)}{(z + Nl + \frac{d_1}{c})^{2k}} = \sum_{d_1 \in \mathbb{Z}/Nc} \varphi(c, d_1) \sum_{l \in \mathbb{Z}} \frac{1}{(z + Nl + \frac{d_1}{c})^{2k}}$$

The Fourier coefficients of the inner sum unwind:

$$\int_0^1 e^{-2\pi i n x / N} \sum_{l \in \mathbb{Z}} \frac{1}{(z + Nl + \frac{d_1}{c})^{2k}} dx = \int_{\mathbb{R}} e^{-2\pi i n x / N} \frac{1}{(x + iy + \frac{d_1}{c})^{2k}} dx$$

As for Eisenstein series for $SL_2(\mathbb{Z})$, this can be evaluated by residues, treating x as a complex variable, replacing the integral along the real line by the limit of integrals $[-R, +R]$ and then over a large arc in either upper or lower half-plane, as $n \leq 0$ or $n > 0$, respectively. The pole at $x = -(iy + \frac{d_1}{c})$ is in the lower half-plane, so for $n \leq 0$ this is 0. For $n > 0$, because the curve is traced clockwise, it is

$$\begin{aligned} -2\pi i \operatorname{Res}_{x=-(iy+\frac{d_1}{c})} e^{-2\pi i n x / N} \frac{1}{(x + iy + \frac{d_1}{c})^{2k}} &= \frac{1}{(2k-1)!} \left(\frac{\partial}{\partial x} \right)^{2k-1} e^{-2\pi i n x / N} \Big|_{x=-(iy+\frac{d_1}{c})} \\ &= -2\pi i \frac{1}{(2k-1)!} (-2\pi i n / N)^{2k-1} \cdot e^{-2\pi i \frac{n}{N} \cdot -(iy+\frac{d_1}{c})} = \frac{(2\pi i)^{2k}}{(2k-1)!} (n/N)^{2k-1} \cdot e^{-2\pi i \frac{n}{N} y} \cdot e^{2\pi i \frac{n}{N} \frac{d_1}{c}} \end{aligned}$$

The exponential in y is as expected. In the sum

$$\sum_{d_1 \in \mathbb{Z}/cN} \varphi(c, d_1) e^{2\pi i \frac{n}{N} \frac{d_1}{c}}$$

the coefficient $\varphi(c, d_1)$ is unchanged under $d_1 \rightarrow d_1 + N$, and the whole sum is stable under this change of variables, so [4]

$$\sum_{d_1 \in \mathbb{Z}/cN} \varphi(c, d_1) e^{2\pi i \frac{n}{N} \frac{d_1}{c}} = \sum_{d_1 \in \mathbb{Z}/cN} \varphi(c, d_1) e^{2\pi i \frac{n}{N} \frac{d_1+N}{c}} = e^{2\pi i \frac{n}{c}} \sum_{d_1 \in \mathbb{Z}/cN} \varphi(c, d_1) e^{2\pi i \frac{n}{N} \frac{d_1}{c}}$$

[4] This part of the computation is a reproof of an instance of the *cancellation lemma*.

Thus, the sum over d_1 is 0 unless $c|n$, and in the latter case

$$\sum_{d_1 \in \mathbb{Z}/cN} \varphi(c, d_1) e^{2\pi i \frac{n}{N} \frac{d_1}{c}} = |c| \cdot \sum_{d_1 \in \mathbb{Z}/N} \varphi(c, d_1) e^{2\pi i \frac{n}{N} \frac{d_1}{c}} \quad (\text{when } c|n)$$

Thus, for $n > 0$, using the bijection $c \rightarrow n/c$ on divisors of n ,

$$\begin{aligned} \int_0^1 e^{-2\pi i n x/N} \tilde{E}_\varphi(x + iy) dx &= \frac{(2\pi i)^{2k}}{(2k-1)! N^{2k-1}} e^{-2\pi n y/N} \sum_{0 \neq c|n} \frac{n^{2k-1}}{c^{2k}} |c| \sum_{d_1 \in \mathbb{Z}/N} \varphi\left(\frac{n}{c}, d_1\right) e^{2\pi i \frac{n}{N} \frac{d_1}{c}} \\ &= \frac{(2\pi i)^{2k}}{(2k-1)! N^{2k-1}} e^{-2\pi n y/N} \sum_{0 \neq c|n} |c|^{2k-1} \sum_{d_1 \in \mathbb{Z}/N} \varphi\left(\frac{n}{c}, d_1\right) e^{2\pi i \frac{n}{N} d_1} \end{aligned}$$

Since the total measure of $[0, N]$ is N , not 1, divide by one further power of N in the $n > 0$ terms to correctly express the Fourier expansion in x , obtaining

$$\tilde{E}_\varphi(x + iy) = \sum_{0 \neq d} \frac{\varphi(0, d)}{d^{2k}} + \frac{(2\pi i)^{2k}}{(2k-1)! N^{2k}} \sum_{n>0} \left(\sum_{0 \neq c|n} |c|^{2k-1} \sum_{d_1 \in \mathbb{Z}/N} \varphi\left(\frac{n}{c}, d_1\right) e^{2\pi i \frac{n}{N} d_1} \right) e^{2\pi i n z/N}$$

[2.3] Back to Eisenstein series for Γ_θ Recall the observation

$$\tilde{E}_\varphi \Big|_{2k} g = \tilde{E}_{\varphi \circ g^{-1}}$$

and the element

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : i\infty \longrightarrow 1$$

From the previous computation, to make a Γ_θ Eisenstein series non-zero at $i\infty$ we want $\varphi(c, d)$ defined mod 2 such that

$$\varphi \circ w = \varphi \quad (\text{so that } \tilde{E}_\varphi|_{2k} w = \tilde{E}_\varphi \text{ with } w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma_\theta)$$

and

$$\tilde{E}_\varphi(i\infty) = \sum_d \frac{\varphi(0, d)}{d^{2k}} \neq 0$$

and

$$\tilde{E}_{\varphi \circ g^{-1}}(i\infty) = \sum_{0 \neq d} \frac{\varphi((0, d)g^{-1})}{d^{2k}} = 0$$

The space of functions on $(\mathbb{Z}/2)^2$ meeting the symmetry condition $\varphi \circ w = \varphi$ is three-dimensional space, spanned by

$$\varphi_1(c, d) = \begin{cases} 1 & (\text{for } (c, d) = (1, 1) \pmod{2}) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\varphi_0(c, d) = \begin{cases} 1 & (\text{for } (c, d) = (0, 0) \pmod{2}) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$\varphi_2(c, d) = \begin{cases} 1 & (\text{for } (c, d) = (0, 1) \text{ or } (1, 0) \pmod{2}) \\ 0 & (\text{otherwise}) \end{cases}$$

The data φ_0 gives an Eisenstein series \tilde{E} with non-zero value at $i\infty$, and is not changed by composition with g^{-1} since $(0,0)$ is stable, so gives a non-zero value at 1, as well.

The data φ_1 has the feature that $\varphi_1(0, d) = 0$, so gives an Eisenstein series \tilde{E} with value 0 at $i\infty$. Composition with g^{-1} is understood by mapping $(1, 1)$ under g , mod 2:

$$(1 \ 1) \cdot g = (1 \ 1) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (1 \ 0) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (0 \ 1) \pmod{2}$$

Thus, the Eisenstein series \tilde{E}_{φ_1} has value at 1, in an appropriate sense, given by

$$(\text{suitable sense of value of } \tilde{E}_{\varphi_1} \text{ at } 1) = \tilde{E}_{\varphi_1} \Big|_{2k} g(i\infty) = \tilde{E}_{0,1}(i\infty) = \sum_{d=1 \pmod{2}} \frac{1}{d^{2k}} \neq 0$$

The value at $i\infty$ of \tilde{E}_{φ_2} is

$$\tilde{E}_{\varphi_2}(i\infty) = \sum_{d=1 \pmod{2}} \frac{1}{d^{2k}} \neq 0$$

To determine the suitable sense of the value of \tilde{E}_{φ_2} at 1, first observe

$$(0 \ 1) \cdot g = (0 \ 1) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (0 \ 1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (1 \ 0) \pmod{2}$$

and

$$(1 \ 0) \cdot g = (1 \ 0) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (1 \ 1) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (1 \ 1) \pmod{2}$$

Thus,

$$(\text{suitable sense of value of } \tilde{E}_{\varphi_2} \text{ at } 1) = \tilde{E}_{\varphi_2} \Big|_{2k} g(i\infty) = \tilde{E}_{1,0}(i\infty) + \tilde{E}_{1,1}(i\infty) = 0$$

In summary, up to normalizing constants, \tilde{E}_{φ_1} is the Eisenstein series *attached to* 1, and \tilde{E}_{φ_2} is the Eisenstein series *attached to* $i\infty$. In both the Fourier expansion of $E_{2k}^{(i\infty)}$ at $i\infty$ and in the Fourier expansion of $E_{2k}^{(1)}$ at 1, the leading non-zero constant is

$$\sum_{d=1 \pmod{2}} \frac{1}{d^{2k}} = 2 \sum_{d \geq 1, d=1 \pmod{2}} \frac{1}{d^{2k}} = 2 \cdot \zeta(2k) \cdot (1 - 2^{-2k})$$

Dividing through by this constant to create values 1 at the corresponding cusps,

$$E_{2k}^{(i\infty)}(z) = 1 + \frac{(2\pi i)^{2k}}{(2k-1)! 2^{2k} 2\zeta(2k)(1-2^{-2k})} \sum_{n>0} \left(\sum_{0 \neq c|n} |c|^{2k-1} \sum_{d_1 \in \mathbb{Z}/2} \varphi_2\left(\frac{n}{c}, d_1\right) e^{2\pi i \frac{c}{2} d_1} \right) e^{2\pi i n z / 2}$$

$$E_{2k}^{(1)}(z) = \frac{(2\pi i)^{2k}}{(2k-1)! 2^{2k} 2\zeta(2k)(1-2^{-2k})} \sum_{n>0} \left(\sum_{0 \neq c|n} |c|^{2k-1} \sum_{d_1 \in \mathbb{Z}/2} \varphi_1\left(\frac{n}{c}, d_1\right) e^{2\pi i \frac{c}{2} d_1} \right) e^{2\pi i n z / 2}$$

These admit minor simplification. In both cases, changing the sign of c does not affect the inner sum, so the 2 in the denominators outside can be dropped while summing over $0 < c|n$. In the case of φ_1 , since $\varphi_1(\frac{n}{c}, d) \neq 0$ only when both $\frac{n}{c}, d$ are *odd*, the inner sum over d_1 gives $(-1)^c$, so

$$E_{2k}^{(1)}(z) = \frac{(2\pi i)^{2k}}{(2k-1)! \zeta(2k)(2^{2k}-1)} \sum_{n>0} \left(\sum_{0 < c|n, \frac{n}{c} \text{ odd}} c^{2k-1} \cdot (-1)^c \right) e^{\pi i n z} \quad (\text{for } \Gamma_\theta)$$

For non-vanishing of $\varphi_2(\frac{n}{c}, d_1)$, either $\frac{n}{c}$ is odd and d is even, or *vice-versa*. For $\frac{n}{c}$ odd and d_1 even, the exponential is 1 for all c . For $\frac{n}{c}$ even and d_1 odd, the exponential is $e^{\pi ic} = (-1)^c$. A formulaic interpolation of the situation is

$$\sum_{d_1 \in \mathbb{Z}/2} \varphi_2\left(\frac{n}{c}, d_1\right) e^{2\pi i \frac{c}{2} d_1} = (-1)^{(1+\frac{n}{c}) \cdot c} = (-1)^{c+n} = (-1)^n \cdot (-1)^c$$

giving

$$E_{2k}^{(i\infty)}(z) = 1 + \frac{(2\pi i)^{2k}}{(2k-1)! \zeta(2k) (2^{2k}-1)} \sum_{n>0} (-1)^n \left(\sum_{0 < c|n} c^{2k-1} \cdot (-1)^c \right) e^{\pi i n z} \quad (\text{for } \Gamma_\theta)$$

[2.3.1] **Remark:** The details of the *outcomes* of the previous computations are less important than the techniques for *doing* the computations. [5]

[2.3.2] **Corollary:** For a holomorphic modular form f of weight $2k \geq 4$ for Γ_θ , with $g \in SL_2(\mathbb{Z})$ such that $g(i\infty) = 1$,

$$f - f(i\infty) \cdot E_{2k}^{(i\infty)} - f|_{2k} g(i\infty) \cdot E_{2k}^{(1)}$$

is a cuspform of weight $2k$.

Proof: The restriction $2k \geq 4$ is for convergence of the Eisenstein series. The Eisenstein series are normalized to have (in a suitable sense) values 1 at their associated cusp and 0 (in a suitable sense) at the other cusp. In that sense, the evaluations of f and $f|_{2k} g$ at $i\infty$ are correct, as above. ///

3. Divisor formula for Γ_θ

Much as the divisor formula for weight $2k$ holomorphic modular forms f for $SL_2(\mathbb{Z})$

$$\frac{\nu_f(i)}{2} + \frac{\nu_f(\rho)}{3} + \nu_f(i\infty) + \sum_{\text{other } z} \nu_f(z) = \frac{2k}{12}$$

with $\nu_f(z)$ the order of vanishing of f at z , gave very strong constraints on spaces of level-one holomorphic modular forms of weight $2k$, in particular proving finite-dimensionality of each such space, an analogous result for Γ_θ is useful, although somewhat less decisively.

For a not-identically-zero holomorphic modular form f of weight $2k$ for Γ_θ , the order of vanishing $\nu_f(\infty)$ is the smallest n_o such that $c_{n_o} \neq 0$ in the Fourier expansion

$$f(x+iy) = \sum_{n \geq 0} c_n e^{\pi i n z}$$

and the order of vanishing $\nu_f(1)$ is the smallest n_o such that $b_{n_o} \neq 0$ in the Fourier expansion

$$f|_{2k}(x+iy) = \sum_{n \geq 0} b_n e^{2\pi i n z}$$

[5] For that matter, these techniques can be further improved by re-expressing Eisenstein series as functions on *adele groups* $GL_2(\mathbb{A})$, completely decoupling the archimedean and the various finite-prime parts of the computation.

[3.0.1] **Theorem:** For a not-identically-zero holomorphic modular form f of weight $2k$ for Γ_θ ,

$$\frac{\nu_f(i)}{2} + \nu_f(i\infty) + \nu_f(1) + \sum_{\text{other } z} \nu_f(z) = \frac{2k}{4}$$

where the sum over *other* z is over mutually Γ_θ -inequivalent points in \mathfrak{H} not Γ_θ equivalent to i .

Proof: As with $SL_2(\mathbb{Z})$, this follows from the *argument principle*, integrating the logarithmic derivative f'/f around the boundary σ of a truncated version of the fundamental domain $F = \{|z| \geq 1, |x| \leq 1\}$ for Γ_θ , and evaluating that integral in another way using the automorphy relations $f(z+2) = f(z)$ and $f(-1/z) = z^{2k}f(z)$. The argument principle gives *essentially*

$$\frac{1}{2\pi i} \int_{\sigma} \frac{f'(z)}{f(z)} dz = (\text{number of zeros inside } \sigma)$$

As for $SL_2(\mathbb{Z})$, there are some technicalities. First, points on the verticals $x = \pm 1$ can be accommodated by slightly indenting the path on $x = -1$ and correspondingly out-denting the path on $x = +1$, so that the modified path does not run right through any such 0. Similarly, any zeros on the arcs cutting off the cusps $i\infty$ and 1 can be dodged by moving the cut-off arc slightly. The point i requires more delicacy, since it is the fixed-point of $z \rightarrow -1/z$, so cannot be dodged so simply, and *half* the residue is picked up.

On the other hand, the automorphy conditions give much cancellation in the integral, as for $SL_2(\mathbb{Z})$. By $f(z+2) = f(z)$, certainly $f'(z+2) = f'(z)$, and since the $x = -1$ and $x = +1$ paths are traced in opposite directions, those parts of the whole integral *cancel* each other completely. The circular arcs from -1 to i and then i to $+1$ are mapped to each other by $z \rightarrow -1/z$, and directions are reversed, so we expect considerable cancellation. However, f and f' are not quite *invariant*: $f(-1/z) = z^{2k} \cdot f(z)$ gives

$$f'(-1/z) \frac{1}{z^2} = 2kz^{2k-1}f(z) + z^{2k}f'(z)$$

so

$$\frac{f'(-1/z)}{f(-1/z)} d(-1/z) = z^2 \cdot \frac{2kz^{2k-1}f(z) + z^{2k}f'(z)}{z^{2k}f(z)} \frac{dz}{z^2} = \left(2kz^{-1} + \frac{f'(z)}{f(z)}\right) dz$$

Thus, the sum of these two arc integrals almost cancels, leaving

$$\begin{aligned} -\frac{1}{2\pi i} \int_0^{\pi/2} 2k(e^{\pi i - it})^{-1} d(e^{\pi i - it}) &= -\frac{1}{2\pi i} 2k \int_0^{\pi/2} (e^{\pi i - it})^{-1} (-i) e^{\pi i - it} dt \\ &= 2k \cdot \frac{\pi/2}{2\pi} = \frac{2k}{4} \end{aligned}$$

The cusps are treated as for $SL_2(\mathbb{Z})$. At $i\infty$ for Γ_θ , the arc cutting off $i\infty$ at some height T captures the *negatives* of the orders of the zeros above that height, and the *negative* of $\nu_f(i\infty)$. Similarly at the cusp 1. Thus, referring only to z in the fundamental domain,

$$\nu_f(i) + \sum_{z \text{ inside path}} \nu_f(z) = \frac{2k}{4} - \nu_f(i\infty) - \nu_f(1) - \sum_{z \text{ outside path}} \nu_f(z)$$

which rearranges to the expected relation. ///

For Γ_θ and holomorphic modular forms:

[3.0.2] **Corollary:** The modular forms of weight 0 are constants. There are no cuspforms of weights 4 or 6. Up to constant multiples, there is a unique cuspform of weight 8, and this cuspform vanishes to first order at both cusps $i\infty$ and 1, and does not vanish on \mathfrak{H} . The weight 2 modular forms are multiples of $E_6^{(i\infty)}/E_4^{(i\infty)}$, and also multiples of $E_6^{(1)}/E_4^{(1)}$. For every weight $2k$ the space of holomorphic modular forms is finite-dimensional.

Proof: From the divisor formula, a weight 0 modular form f cannot vanish unless it vanishes identically, so $f - f(z_o)$ is identically 0, for any $z_o \in \mathfrak{H}$.

Since $2k/4 < 2$ for $2k = 4, 6$, and since a cuspform vanishes to order at least 1 at both cusps $i\infty$ and 1, there can be no cuspforms of weights 4 or 6.

Similarly, at weight $2k = 8$, we have $2k/4 = 2$, so the vanishing of a cuspform at the two cusps, would leave no room for any other vanishing. For existence, the *product* f of the two weight 4 Eisenstein series $E_4^{(i\infty)}$ and $E_4^{(1)}$ vanishes at both cusps, and is not identically 0. Thus, given a cuspform F of weight $2k \geq 8$, the quotient F/f is a modular form of weight $2k - 8$.

The general finite-dimensionality follows essentially by induction. Given a holomorphic modular form F of weight $2k \geq 8$, by subtracting multiples of the two Eisenstein series, without loss of generality F is a cuspform. Divide by the unique weight 8 cuspform so that F/f is of weight $2k - 8$. Thus, it suffices to prove finite-dimensionality for weights 0, 2, 4, 6, already accomplished for weights 0, 4, 6. The space of weight 2 modular forms F is mapped linearly to weight 6 modular forms by $F \rightarrow F \cdot E_4^{(i\infty)}$. Since $E_4^{(i\infty)}$ vanishes at the cusp 1, and F vanishes at i , these are the only zeros of this product. The Eisenstein series $E_6^{(i\infty)}$ has the same vanishing, so

$$\text{weight 2 modular form for } \Gamma_\theta = (\text{multiple of}) \frac{E_6^{(i\infty)}}{E_4^{(i\infty)}}$$

A similar argument applies with $E_6^{(1)}/E_4^{(1)}$.

///

4. Harmonic theta series

Let f be homogeneous, harmonic, of degree d , on \mathbb{R}^n . The associated *harmonic theta series* is

$$\theta_f(z) = \sum_{v \in \mathbb{Z}^n} f(v) e^{\pi i |v|^2 z}$$

For *odd* n this has more complicated behavior than for n *even*. The behavior is clearest for n divisible by 8, for reasons visible in the computation below, and we soon specialize to that case.

[4.1] **Convergence** It is not surprising that the exponential decay of

$$v \longrightarrow |e^{\pi i |v|^2 z}| = e^{-\pi |v|^2 y}$$

assures convergence of the series defining $\theta_f(z)$. Specifically, let $\nu(N)$ be the number of lattice points $v \in \mathbb{R}^n$ such that $|v|^2 = N$. First, since \sqrt{n} is the length of diagonal of an n -dimensional cube, and since the number of lattice points inside a large ball is at most the number of unit cubes inside a ball with radius larger by \sqrt{n} , $\nu(M)$ can be estimated by

$$\begin{aligned} \nu(M) &\leq \#\{v : |v|^2 \leq \sqrt{M}\} \leq \#\{\text{unit cubes inside } \{v : |v|^2 \leq \sqrt{M} + \sqrt{n}\}\} \\ &\leq \text{vol}(\text{ball of radius } \sqrt{M} + \sqrt{n}) \ll M^{n/2} \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{|v|^2 \leq N} |f(v) e^{\pi i |v|^2 z}| &\leq \sum_{N_o \leq M \leq N} M^{n/2} e^{-\pi M y} \ll_n \sum_{N_o \leq M \leq N} e^{M/2} e^{-\pi M y} \\ &\ll_n \sum_{N_o \leq M \leq N} e^{-(\pi y - \frac{1}{2})M} \leq \frac{e^{-(\pi y - \frac{1}{2})N_o}}{1 - e^{-(\pi y - \frac{1}{2})M}} \end{aligned}$$

This gives absolute convergence uniform on compacts, and also proves *boundedness* of $\theta_f(z)$ as $y \rightarrow +\infty$.

[4.2] Hecke's identity, Poisson summation, and inversion $z \rightarrow -1/z$ Recall Hecke's identity [6]

$$\left(f(x) e^{-\pi|x|^2} \right)^\wedge(\xi) = i^{-d} \cdot f(\xi) e^{-\pi|\xi|^2} \quad (\text{for } f \text{ degree } d \text{ harmonic})$$

Fourier transform behaves well with respect to dilation: for $y > 0$, letting $(F \circ y)(x) = F(yx)$, a direct computation gives

$$\begin{aligned} (F \circ y)^\wedge(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot x} F(yx) dx = \int_{\mathbb{R}^n} e^{-2\pi i \xi \cdot \frac{x}{y}} F(x) y^{-8n} dx \\ &= y^{-n} \int_{\mathbb{R}^n} e^{-2\pi i \frac{\xi}{y} \cdot x} F(x) dx = y^{-n} \left(\widehat{F} \circ \frac{1}{y} \right) (\xi) \end{aligned}$$

With y replaced by \sqrt{y} , invoking Poisson summation

$$\sum_{v \in \mathbb{Z}^n} \varphi(v) = \sum_{v \in \mathbb{Z}^n} \widehat{\varphi}(v)$$

compute

$$\begin{aligned} \theta_f(iy) &= \sum_{v \in \mathbb{Z}^n} f(v) e^{-\pi|v|^2 y} = \frac{1}{\sqrt{y}^d} \sum_{v \in \mathbb{Z}^n} f(v \cdot \sqrt{y}) e^{-\pi|v \cdot \sqrt{y}|^2} = \frac{1}{\sqrt{y}^d} \cdot \frac{1}{\sqrt{y}^n} \cdot i^{-d} \cdot \sum_{v \in \mathbb{Z}^n} f\left(\frac{v}{\sqrt{y}}\right) e^{-\pi|v/\sqrt{y}|^2} \\ &= \frac{1}{(iy)^{d+\frac{n}{2}}} \sum_{v \in \mathbb{Z}^n} f(v) e^{-\pi|v|^2/y} = \frac{1}{(iy)^{d+\frac{n}{2}}} \cdot \theta_f(-1/iy) \end{aligned}$$

[4.2.1] **Remark:** Here the complication of odd n is visible, namely, choosing a branch of square root. This is an *essential* complication, as it turns out that half-integral weight modular forms have very different behavior from integral-weight.

Thus, avoiding the half-integral weight complication, and a few further comparatively minor complications, replace n by $8n$. By the identity principle,

$$\theta_f(z) = \frac{1}{z^{d+4n}} \cdot \theta_f(-1/z)$$

This is the main issue in proving that $\theta_f(z)$ is a weight $d + 4n$ modular form for Γ_θ . The lengths-squared $|v|^2$ are all integers, so the exponentials $e^{\pi i|v|^2 z}$ are all invariant under the translation $z \rightarrow z + 2$. We showed that inversion and this translation generate Γ_θ .

[4.3] **Behavior at cusp $i\infty$** Above, we showed that $\theta_f(z)$ is bounded as $y \rightarrow +\infty$. The defining expression for θ_f shows that its Fourier expansion at the cusp $i\infty$ has no negative-index coefficients. Indeed, for $d > 0$, the 0 th Fourier coefficient at $i\infty$ is $f(0) = 0$, so $\theta_f(i\infty) = 0$.

[4.4] **The other cusp 1** Checking the behavior of θ_f at the cusp 1 requires verifying that $(\theta_f|_{d+4n} g)(z)$ is bounded as $z \rightarrow +i\infty$, for $g \in SL_2(\mathbb{Z})$ such that $g(i\infty) = 1$.

[6] Recall that Hecke's identity is nicely proven by invoking the *irreducibility* of the space H_d of harmonic, degree d homogeneous polynomials on \mathbb{R}^n , that is, there is no $O(n)$ -stable subspace of H_d except $\{0\}$ and H_d itself. Since $O(n)$ commutes with Fourier transform, verifying Hecke's identity for a single (not-identically-zero) $f \in H_d$ verifies it for all.

That is, the question is *not* simply about the values of $\theta_f(z)$ as z goes to 1 inside the fundamental domain, but about $\theta_f(z)$ altered by the weight $d + 4n$ action.

With

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and noting the associativity

$$\theta_f|_{d+4n}g = \left(\theta_f|_{d+4n} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)|_{d+4n} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

first compute

$$\left(\theta_f|_{d+4n} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)(z) = \sum_v f(v) e^{\pi i |v|^2 (z+1)} = \sum_v f(v) e^{\pi i |v|^2} e^{\pi i |v|^2 z} = \sum_v f(v) (-1)^{|v|^2} e^{\pi i |v|^2 z}$$

That is, the summands are twisted by powers of -1 , complicating application of Poisson summation to achieve the further effect of inversion $z \rightarrow -1/z$. Rewrite the sum as a sum over translates of lattices on each of which $e^{\pi i |v|^2} = (-1)^{|v|^2}$ is constant: replace v by $w + 2v$ with $v \in \mathbb{Z}^{8n}$ and $w \in \mathbb{Z}^{8n} \bmod 2$:

$$|w + 2v|^2 = |w|^2 + 2\langle w, 2v \rangle + 4|v|^2 \in |w|^2 + 2\mathbb{Z}$$

so

$$\theta_f(z+1) = \sum_v f(v) e^{\pi i |v|^2} e^{\pi i |v|^2 z} = \sum_{w \in \mathbb{Z}^{8n}/2} e^{\pi i |w|^2} \sum_{v \in \mathbb{Z}^{8n}} f(2v+w) e^{\pi i |2v+w|^2 z}$$

For each w , we will apply Poisson summation to the corresponding inner sum. Recall the behavior of Fourier transform on \mathbb{R}^m under affine transformations $x \rightarrow ax + b$: by direct computation,

$$\begin{aligned} F(ax+b)^\wedge(\xi) &= \int_{\mathbb{R}^m} e^{-2\pi i \langle \xi, x \rangle} F(ax+b) dx = \frac{1}{|a|^m} \int_{\mathbb{R}^m} e^{-2\pi i \langle \xi, \frac{x}{a} \rangle} F(x+b) dx \\ &= \frac{1}{|a|^m} \int_{\mathbb{R}^m} e^{-2\pi i \langle \frac{\xi}{a}, x \rangle} F(x+b) dx = \frac{1}{|a|^m} \int_{\mathbb{R}^m} e^{-2\pi i \langle \frac{\xi}{a}, x-b \rangle} F(x) dx \\ &= \frac{1}{|a|^m} e^{2\pi i \langle \xi, \frac{b}{a} \rangle} \int_{\mathbb{R}^m} e^{-2\pi i \langle \frac{\xi}{a}, x \rangle} F(x) dx = \frac{1}{|a|^m} e^{2\pi i \langle \xi, \frac{b}{a} \rangle} \cdot \widehat{F}\left(\frac{\xi}{a}\right) \end{aligned}$$

Applying this in Poisson summation, with $F(x) = f(x) e^{-\pi |x|^2}$ and $a = 2\sqrt{y}$ and $b = \sqrt{y}w$, via Hecke's identity the w^{th} inner sum at $z = iy$ is

$$\begin{aligned} \sum_{v \in \mathbb{Z}^{8n}} f(2v+w) e^{-\pi |2v+w|^2 y} &= \frac{1}{\sqrt{y}^d} \sum_{v \in \mathbb{Z}^{8n}} f(2\sqrt{y}v + \sqrt{y}w) e^{-\pi |2\sqrt{y}v + \sqrt{y}w|^2} \\ &= \frac{1}{\sqrt{y}^d} \cdot \frac{1}{(2\sqrt{y})^{8n}} \cdot i^{-d} \cdot \sum_{v \in \mathbb{Z}^{8n}} e^{2\pi i \langle v, \frac{\sqrt{y}w}{2\sqrt{y}} \rangle} f\left(\frac{v}{2\sqrt{y}}\right) e^{-\pi |v|^2 / 4y} \\ &= \frac{1}{\sqrt{y}^d} \cdot \frac{1}{(2\sqrt{y})^{8n}} \cdot i^{-d} \cdot \frac{1}{(2\sqrt{y})^d} \sum_{v \in \mathbb{Z}^{8n}} e^{2\pi i \langle v, \frac{w}{2} \rangle} f(v) e^{-\pi |v|^2 / 4y} \\ &= \frac{1}{(iy)^{d+4n}} \cdot \frac{1}{2^{8n+d}} \cdot \sum_{v \in \mathbb{Z}^{8n}} e^{\pi i \langle v, w \rangle} f(v) e^{-\pi |v|^2 / 4y} \end{aligned}$$

Summing over $w \in \mathbb{Z}^{8n}/2$ gives

$$\theta_f(iy+1) = \frac{1}{(iy)^{4n+d} 2^{8n+d}} \sum_{v \in \mathbb{Z}^{8n}} \left(\sum_{w \in \mathbb{Z}^{8n}/2} e^{\pi i |w|^2} \cdot e^{\pi i \langle v, w \rangle} \right) f(v) e^{\pi i \frac{|v|^2}{4} (-1/iy)}$$

The identity principle gives the same identity for $z \in \mathfrak{H}$ in place of iy :

$$\theta_f(z+1) = \frac{1}{z^{4n+d} 2^{8n+d}} \sum_{v \in \mathbb{Z}^{8n}} \left(\sum_{w \in \mathbb{Z}^{8n}/2} e^{\pi i |w|^2} \cdot e^{\pi i \langle v, w \rangle} \right) f(v) e^{\pi i \frac{|v|^2}{4} (-1/z)}$$

Recall that $d \in 2\mathbb{Z}$ to avoid identical vanishing of θ_f . Replacing z by $-1/z$ and rearranging slightly,

$$z^{-(4n+d)} \cdot \theta_f\left(\frac{-1}{z} + 1\right) = \frac{1}{2^{8n+d}} \sum_{v \in \mathbb{Z}^{8n}} \left(\sum_{w \in \mathbb{Z}^{8n}/2} e^{\pi i |w|^2} \cdot e^{\pi i \langle v, w \rangle} \right) f(v) e^{\pi i \frac{|v|^2}{4} z}$$

Without concern about further simplification, this exhibits the Fourier expansion of θ_f at 1, and there are no negative-index terms. In particular, as $y \rightarrow +\infty$, elementary estimates prove this is *bounded*, completing the proof that θ_f is a modular form of weight $d + 4n$ for Γ_θ .

For degree $d > 0$, the 0^{th} Fourier coefficient is a multiple of $f(0) = 0$, so in this case θ_f is a *cusppform*.

For degree $d = 0$ and $f = 1$, the 0^{th} Fourier coefficient of $\theta_f|_{d+4n} g$ is

$$\frac{1}{2^{8n}} \sum_{w \in \mathbb{Z}^{8n}/2} e^{\pi i |w|^2} = \frac{1}{2^{8n}} \left(\sum_{w \in \mathbb{Z}/2} e^{\pi i w^2} \right)^{8n} = \frac{1}{2^{8n}} (1-1)^{8n} = 0$$

That is, θ_f *vanishes* at the cusp 1, for *all* f .

[4.4.1] Remark: It is certainly not obvious from the original expression for $\theta_f(z)$ that for $d = 0$ and $f = 1$ the theta series vanishes (in a suitable sense) at cusp 1.

5. Sums of squares

Let $\nu_{8n}(N)$ be the number of ways to express N as a sum of $8n$ squares of integers. Slightly subverting the notation θ_f , let

$$\theta_{8n}(z) = \sum_{v \in \mathbb{Z}^{8n}} e^{\pi i |v|^2 z} = \sum_{0 \leq N} \nu_{8n}(N) e^{\pi i N z}$$

This is a weight $4n$ holomorphic modular form for Γ_θ , and the previous section shows that it vanishes at the cusp 1. The value of $\theta_{8n}(z)$ at the cusp $i\infty$ is the 0^{th} Fourier coefficient, namely, 1.

[5.1] Eight squares The *divisor formula* showed that there are no cusppforms of weight 4 for Γ_θ . Thus, $\theta_8(z)$ is the Eisenstein series $E_4^{(i\infty)}(z)$ normalized so as to take value 1 at cusp $i\infty$ and 0 at cusp 1. Thus,

$$\begin{aligned} \nu_8(N) &= N^{\text{th}} \text{ Fourier coefficient of } E_4^{(i\infty)} = \frac{(2\pi i)^4}{3! \zeta(4)(2^4-1)} (-1)^N \sum_{0 < c|N} c^3 \cdot (-1)^c \\ &= \frac{16\pi^4}{6 \cdot \frac{\pi^4}{90} \cdot 15} (-1)^N \sum_{0 < c|N} c^3 \cdot (-1)^c = 16 \cdot (-1)^N \sum_{0 < c|N} c^3 \cdot (-1)^c \end{aligned}$$

In fact, the value of the constant $\frac{(2\pi i)^4}{3! \zeta(4)(2^4-1)}$ could be *determined* by evaluating $\nu_8(1)$.

[5.2] $8n$ squares For $8n > 8$, the weight $4n$ is above the range where there are no cusppforms for Γ_θ , so in general we invoke Hecke's estimate on Fourier coefficients of cusppforms, and can only say

$$\begin{aligned} \nu_{8n}(N) &= N^{\text{th}} \text{ Fourier coefficient of } \left(E_{4n}^{(i\infty)} + \text{cuspform of weight } 4n \right) \\ &= \frac{(2\pi i)^{4n}}{(4n-1)! \zeta(4n) (2^{4n}-1)} (-1)^N \left(\sum_{0 < c|N} c^{4n-1} \cdot (-1)^c \right) + O(N^{\frac{4n}{2n}}) \end{aligned}$$

[5.2.1] **Corollary:** For every $8n \geq 8$, there is a constant $C_n > 0$, such that every sufficiently large $N \geq 1$ can be represented in at least $C_n \cdot N^{4n-1}$ ways as a sum of $8n$ squares.

Proof: The signed sum of powers of divisors function is *weakly multiplicative*, so it suffices to give a lower bound for prime powers p^e : first, for odd prime p ,

$$(-1)^{p^e} \sum_{0 < c|p^e} c^{4n-1} (-1)^c = (p^e)^{4n-1} - (p^{e-1})^{4n-1} + \dots + (-1)^{p^e} \geq (p^e)^{4n-1} \cdot \left(\frac{1}{1 + p^{-(4n-1)}} \right)$$

For $p = 2$,

$$(-1)^{p^e} \sum_{0 < c|p^e} c^{4n-1} (-1)^c = (p^e)^{4n-1} + (p^{e-1})^{4n-1} + \dots + 1 \geq \frac{(p^e)^{4n-1}}{1 - p^{-(4n-1)}} \geq \frac{(p^e)^{4n-1}}{1 + p^{-(4n-1)}}$$

The product of $1/(1 + p^{-(4n-1)})$ over all primes is convergent to a finite non-zero constant B_n . Thus,

$$(-1)^N \left(\sum_{0 < c|N} c^{4n-1} \cdot (-1)^c \right) \geq B_n \cdot N^{4n-1}$$

Further adjustment by the leading constant does not alter the qualitative conclusion. ///

[5.2.2] **Corollary:** A prime $p > 2$ can be written as a sum of 8 squares in $16(p^3 + 1)$ ways. ///

6. Equidistribution of integer points projected to the sphere

Taking the dimension $8n$ to avoid various complications already visible in the discussion of harmonic theta series, let

$$X_N = \{v \in \mathbb{Z}^{8n} : |v|^2 = N\}$$

[6.0.1] **Theorem:** The projections $\{\frac{v}{|v|} : v \in X_N\}$ of X_N to the unit sphere are *equidistributed*, in the sense that for every $f \in C^\infty(S^{8n-1})$

$$\lim_{N \rightarrow \infty} \frac{1}{\#X_N} \sum_{v \in X_N} f\left(\frac{v}{|v|}\right) = \int_{S^{8n-1}} f$$

for *some* normalization of the total measure of S^{8n-1} .

Proof: We prove another instance of *Weyl's equidistribution criterion*, for functions on spheres:

[6.0.2] **Claim:** A sequence Y_N of subsets of S^{n-1} is *equidistributed* if and only if, for every f harmonic, homogeneous of degree $d > 0$,

$$\lim_{N \rightarrow \infty} \frac{1}{\#Y_N} \sum_{v \in Y_N} f(v) = 0$$

Proof: (of claim) Let H_d be the space of harmonic, homogenous degree d polynomials on S^{n-1} . For the easy direction, certainly functions $f \in H_d$ are smooth, and for $d > 0$ are orthogonal to constants, so equidistribution implies

$$\lim_{N \rightarrow \infty} \frac{1}{\#Y_N} \sum_{v \in Y_N} f(v) = 0$$

For the more substantive implication, from earlier discussion of Sobolev imbedding theorems on spheres, a smooth function F on S^{n-1} has a Fourier expansion $\sum_d F_d$, with F_d the orthogonal projection of F to H_d , converging absolutely to $F(x)$ uniformly in $x \in S^{n-1}$.

From the discussion of harmonic analysis on spheres,

$$\frac{\sup_{x \in S^{n-1}} |F_d(x)|}{|F_d|_{L^2(S^{n-1})}} \ll d^{n-2} \quad (\text{for } F_d \in H_d)$$

with implied constant depending on the dimension n and the choice of total measure for S^{n-1} , but not otherwise on d . Thus,

$$\left| \frac{1}{\#Y_N} \sum_{v \in Y_N} F_d(v) \right| \ll d^{n-2} \cdot |F_d|_{L^2(S^{n-1})} \quad (\text{for } F_d \in H_d)$$

and

$$\frac{1}{\#Y_N} \sum_{v \in Y_N} F(v) = \frac{1}{\#Y_N} \sum_{v \in Y_N} \sum_d F_d(v) = \sum_d \left(\frac{1}{\#Y_N} \sum_{v \in Y_N} F_d(v) \right)$$

Using $F_0 = \int_{S^{n-1}} F$, for any cut-off b , estimate $\frac{1}{\#Y_N} \sum_{v \in Y_N} F(v) - \int_{S^{n-1}} F$ by

$$\sum_{d>0} \frac{1}{\#Y_N} \left| \sum_{v \in Y_N} F_d(v) \right| \ll \sum_{0<d \leq b} \frac{1}{\#Y_N} \left| \sum_{v \in Y_N} F_d(v) \right| + \sum_{d>b} |F_d|_{L^2} \cdot d^{n-2}$$

For F smooth with $F = \sum_d F_d$ in an L^2 -sense, with Δ^S the Laplacian on the sphere with eigenvalue $\lambda_d = -d(d+n-2)$ on H_d , $(\Delta^S)^\ell F$ is still smooth, and

$$H_d\text{-component of } (\Delta^S)^\ell F = (\lambda_d)^\ell \cdot H_d\text{-component of } F = (-d(d+n-2))^\ell \cdot H_d\text{-component of } F$$

Since $(\Delta^S)^\ell F$ is in $L^2(S^{n-1})$ for all ℓ , by Plancherel applied to $(\Delta^S)^\ell F$,

$$\sum_d \left| (-d(d+n-2))^\ell \right|^2 \cdot |F_d|_{L^2}^2 = |(\Delta^S)^\ell F|^2 < \infty$$

Thus, given $\varepsilon > 0$ there is large-enough b so that

$$\left(\sum_{d>b} |F_d|_{L^2} \cdot d^{n-2} \right)^2 = \left(\sum_{d>b} |F_d|_{L^2} \cdot d^{n-1} \cdot \frac{1}{d} \right)^2 \leq \left(\sum_{d>b} |F_d|_{L^2}^2 \cdot d^{2(n-1)} \right) \cdot \left(\sum_{d>b} \frac{1}{d^2} \right) < \varepsilon^2$$

via Cauchy-Schwarz-Bunyakowsky. With such b , since $\frac{1}{\#Y_N} \sum_{v \in Y_N} F_d(v) \rightarrow 0$ for each fixed $d > 0$, and since there are only finitely-many d with $0 < d \leq b$, for large-enough N

$$\sum_{0<d \leq b} \frac{1}{\#Y_N} \left| \sum_{v \in Y_N} F_d(v) \right| < \varepsilon$$

Thus,

$$\left| \frac{1}{\#Y_N} \sum_{v \in Y_N} F(v) - F_0 \right| < 2\varepsilon$$

That is, $\frac{1}{\#Y_N} \sum_{v \in Y_N} F(v) \rightarrow \int_{S^{n-1}} F$, and Weyl's criterion suffices for equidistribution. ///

Now return to the proof of equidistribution, invoking Weyl's criterion. Given $f \in H_d$ on S^{8n-1} with $d > 0$, we must show that

$$0 = \lim_N \frac{1}{\#X_N} \sum_{v \in X_N} f\left(\frac{v}{|v|}\right) = \lim_N \frac{1}{N^{\frac{d}{2}}} \frac{1}{\#X_N} \sum_{v \in X_N} f(v)$$

using the homogeneity of f , since $|v| = \sqrt{N}$ for $v \in X_N$. From computation of Fourier coefficients of the Eisenstein series $E_{4n}^{(i\infty)}$ and Hecke's estimate, we saw that

$$\#X_N \gg_m N^{4n-1}$$

Of course,

$$\sum_{v \in X_N} f(v) = N^{\text{th}} \text{ Fourier coefficient of harmonic theta series } \theta_f$$

In dimension $8n$ with degree d , θ_f is of weight $4n + d$ for Γ_θ , so by Hecke's estimate

$$\left| \sum_{v \in X_N} f(v) \right| \ll_f N^{\frac{4n+d}{2}}$$

and

$$\frac{1}{\#X_N} \sum_{v \in X_N} f\left(\frac{v}{|v|}\right) \ll \frac{1}{N^{4n-1}} \cdot \frac{1}{N^{\frac{d}{2}}} \cdot N^{\frac{4n+d}{2}} = \frac{1}{N^{2n-1}} \rightarrow 0 \quad (\text{for } 2n - 1 > 0)$$

meeting Weyl's equidistribution criterion. ///

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